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THE COMPLEXITY OF σ-DISCRETELY DECOMPOSABLE FAMILIES IN UNIFORM SPACES

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Abstract: We analyze the σ-discretely decomposable families in a uniform space. Particularly, we construct examples of spaces with prescribed relation of σ-discretely decomposable and σ-discrete families.

Key words: Reflective and coreflective subcategory of Unif, σ-discretely decomposable family.

Classification: 54E15, 54H05

Generalizing the metric notion of R.W. Hansell, a family $\{X_\alpha \mid \alpha \in I\}$ of subsets of a uniform space X is called σ-discretely decomposable (abbr. σ-d.d.) if any X_α can be expressed as a union of countably many sets, $X_\alpha = \bigcup_{n=1}^{\infty} X_\alpha^n$, in such a way that $\{X_\alpha^n \mid \alpha \in I\}$ is uniformly discrete in X for any $n \in N$. The notion of σ-d.d. family was extensively studied by several authors (see [F], [FH$_1$], [FH$_2$], [H$_1$], [H$_2$], [Ho], [KP], [PP]) and has occuppied an important place in the nonseparable descriptive theory of sets.

Naturally, the σ-d.d. families in a uniform space may be essentially more complex than those of a metric space. We exhibit in the present paper certain phenomena which one may find useful to be aware of when one is to deal with σ-d.d. families. The main contribution of our note is, we think, the counterexamples.

The contents begins with the examination of a subclass of Unif consisting of the spaces satisfying the condition: Any σ-d.d. family of σ-d.d. families is again σ-d.d. family. We call the subclass Decomp. We show among others that Decomp constitutes a large (proper) subclass of Unif, containing e.g. metric spaces, kinds of "locally fine" spaces and spaces which are themselves σ-d.d., and we test Decomp as regard the formation of categorical operations - products, quotions, etc.

In the second part we show that even relatively simple spaces may have considerably richer structure of σ-d.d. families than that of σ-discrete ones. The examples might be of an intuitive value for the nonseparable descriptive theory of sets (as indicated in [Ho]), or elsewhere in the realm of Unif.

Let us mention that this paper overlaps slightly with the paper [FH_1] where the authors persue the questions of nonseparable descriptive theory of sets in Unif and independently encounter one or two analogical problems.

§ 1. The class Decomp. Miscellaneous. Throughout the paper, the word space always means uniform space and the word discrete uniformly discrete.

Definition 1.1: A space X is said to be in the class Decomp if the following condition is satisfied. Suppose that the families $\{A_\alpha \mid \alpha \in I\}$ and $\{B_\alpha^\beta \mid \beta \in J_\alpha\}$ are σ-d.d. in X and suppose that $A_\alpha = \bigcup_{\beta \in J_\alpha} B_\alpha^\beta$ for any $\alpha \in I$. Then the family $B_\alpha^\beta \mid \alpha \in I, \{\beta \in J_\alpha\}$ is σ-d.d. in X, too.

We can restate the latter definition as follows.

Statement 1.1: A space X belongs to Decomp iff any discrete family of discrete families is σ-d.d.

Proof is easy.

Statement 1.2: Decomp is closed under the formation of subspaces and sums in Unif.

Proof is evident.

Statement 1.3: Any metric space belongs to Decomp. Thus, any uniform space is a subspace of a product of spaces from Decomp and so the epireflective hull of Decomp in Unif is the entire Unif.

Proof: Any discrete family of discrete families in a metric space is σ-discrete (and therefore σ-d.d.).

Definition 1.2: Following J. Isbell (see [I]), a space is called locally fine (abbr. LF) if the following condition is satisfied. If $\mathcal{X} = \{X_\alpha | \alpha \in I\}$ and $\mathcal{X}_\alpha = \{Y_\alpha^\beta | \beta \in J_\alpha\}$ are uniform coverings then so is the covering $\{X_\alpha \cap Y_\alpha^\beta | \alpha \in I, \beta \in J_\beta\}$. A space is called finite-dimensionally locally fine (abbr. FDLF) if the latter condition holds for $\mathcal{X}$ finitely dimensional and all $\mathcal{X}_\alpha$ of at most dimension n for an $n \in N$.

Statement 1.4: Any FDLF space belongs to Decomp.

Proof: As the discreteness can be realized by one-dimensional uniform coverings only, we obtain that in FDLF any discrete family of discrete families is discrete.

We do not know if there is the largest coreflective category in Unif contained in Decomp. Apparently FDLF spaces may be a candidate. Of course, Decomp itself is not coreflective as the next statement establishes. Prior to that, we introduce an important class of spaces which is going to appear throughout the paper.

Definition 1.3: A space X is called σ-d.d. in itself if $\{x \mid x \in X\}$ is σ-d.d.

Note that it would not change the latter definition if we said σ-discrete instead of σ-d.d. .

Statement 1.5: Any σ-d.d. in itself space belongs to Decomp. Moreover, any uniform space is a quotient of a σ-d.d. in itself space and so the coreflective hull of Decomp in Unif is the entire Unif.

Proof: The first claim is evident. As for the second, the reader is invited to consult the book [I] (p. 52) or [Č] (p. 699) and check that the construction presented here has the desired properties.

The next statements on Decomp concern the closedness under the formation of products.

Statement 1.6: There exists a space X, $X \in$ Decomp, such that, for a discrete space D, $X \times D$ does not belong to Decomp.

Proof: Take a set X with card $X > 2^{\omega_1}$. Endow X with the uniformity whose base is formed by the coverings of the type $\mathfrak{X}_Y = \{x \mid x \in X - Y\} \cup \{Y\}$, card$(X-Y) = \omega_1$. Of course, $X \in$ Decomp. Now, we claim that $X \times D \notin$ Decomp provided card $D \geq 2^X$. Indeed, take for any $d \in D$ a set A_d, $A_d \subset X \times \{d\}$, card $A_d = \omega_1$ and do it in such a manner that for any set Y with card Y = $= \omega_1$ we have a $d \in D$ such that $Y = A_d$. Obviously, $\{A_d \mid d \in D\}$ is a discrete family and any A_d is discrete when considered as a family of singletons. On the other hand, $A = \bigcup_{d \in D} A_d$ is not σ-d.d. when considered as a family of singletons. The proof is complete.

Statement 1.7: If D is an uncountable discrete space then D^{ω_1} does not belong to Decomp. A corollary: The largest epireflective subcategory of Unif contained in Decomp is the class of spaces which possess a base consisting of countable coverings.

Proof: Obviously, the second part of the statement follows easily from the first one because if a space does not have such a base it possesses an uncountable discrete set. We will sketch the proof of the first part for the space D with card $D = \omega_1$. Set $A_{d_0} = \{d_0\} \times D^{\omega_1 - \{0\}} \subset D^{\omega_1}$ and take the family $\{A_{d_0} \mid d_0 \in D\}$. Obviously, $\{A_{d_0} \mid d_0 \in D\}$ is discrete in D^{ω_1}. Choose a one-to-one mapping φ between D and the set of all finite subsets of $\omega_1 - \{0\}$. Now, put $A_{d_0} = \cup B^{\beta}_{d_0}$ where β goes over all $(d_{\alpha_1}, d_{\alpha_2}, \dots, d_{\alpha_n})$ such that $(\alpha_1, \alpha_2, \dots, \alpha_n) = \varphi(d_0)$ and we set

$$B_{d_0}^{(d_{\alpha_1}, d_{\alpha_2}, \dots, d_{\alpha_n})} = \{d_0\} \times \{d_{\alpha_1}\} \times \dots \times \{d_{\alpha_n}\} \times D^{\omega_1 - \{0\} - \varphi(d_0)} .$$

It is somewhat technical but essentially not difficult to show that the full family $\{B^{\beta}_{d_0}\}$ is not σ-d.d. We allow us to leave it to the reader.

§ 2. <u>Examples of uniform spaces with prescribed relations between σ-d.d. families and σ-discrete families</u>.

Here we produce three examples demontrating basic features that are to be realized if one strives for deeper understanding of the structure of σ-d.d. families. The examples concern the measure of dependence of σ-d.d. families and the σ-discrete ches.

Example 2.1: There is a σ-d.d. in itself space possessing a discrete family of discrete families which is not σ-discrete.

Proof: To start with, we form an auxiliary space Y as follows. We take a partition of ω_1 into ω_1 sets, each of them of cardinality ω_0. Thus, we have $\omega_1 = \bigvee_{\alpha \in \omega_1} P_\alpha$, card $P_\alpha = \omega_0$. The space Y will be carried by the set $\omega_0 \times \omega_1$ and the uniformity of Y will be determined by requiring the coverings of the form $\mathcal{X}_F = \{(n,\sigma) \mid n \in F,\ F \text{ finite},\ \sigma \in \omega_1\} \cup \cup \{(\omega_0 - F) \times \omega_1\}$ to constitute a base. Now, take a discrete uniform space D, card D = ω_1, and put $X' = Y \times D$ (endowed with the product uniformity). Add the coverings of the form $\mathcal{Y}_F = \{(\omega_0 \times P_\alpha) \times k \mid k \in F,\ F \text{ finite},\ \alpha \in \omega_1\} \cup \{Y \times (D - F)\}$ to the space X' and denote so obtained space by X. We are ready to prove that X has the property in question.

First, fixing an $n \in \omega_0$, the set $(n \times \omega_1) \times D$ is discrete in itself and therefore X is σ-d.d. in itself. On the other hand, the family $\mathcal{F} = \{(\omega_0 \times P_\alpha) \times k \mid \alpha \in \omega_1, k \in D\}$ can be obtained as a discrete family of discrete families but $\mathcal{F}$ is not σ-discrete. Indeed, if we take a discrete subfamily $\mathcal{F}'$ of $\mathcal{F}$ we see easily that there exists a finite set F, $F \subset D$, such that the set $S_k = \{\alpha \mid (\omega_0 \times P_\alpha) \times k \in \mathcal{F}'\}$ is at most a singleton for all $k \in D - F$. The latter observation together with the fact that card D = ω_1 violates the family $\mathcal{F}$ being σ-discrete.

The following example completes the results of $[F_1]$.

Example 2.2: There exists a non-discrete σ-d.d. in itself space X with the property: If ρ is a pseudometric on X and if ρ creates less σ-discrete families than the uniformity of X then ρ is uniformly continuous.

Proof: The space X will be carried by the set $\bigvee_{n=1}^{\infty} S_n$ with all S_n, $n \in N$ of cardinality ω_1. The base of X will consist of all partitions $\mathcal{P}$ of X such that, for a finite $F_{\mathcal{P}}$, the restriction of $\mathcal{P}$ on $X - \bigvee_{n \in F_{\mathcal{P}}} S_n$ is a countable partition and the restriction of $\mathcal{P}$ on $\bigvee_{n \in F_{\mathcal{P}}} S_n$ is discrete. Obviously, X is σ-d.d. in itself. Suppose we have a pseudometric ρ contradicting the desired properties of X. Then there exists an $\varepsilon > 0$ and an infinite $\omega, \omega \subset \omega_0$ such that, for any $n \in \omega$, we can find an uncountable ε-discrete family $D_n = \{d_n^{\alpha} | \alpha \in \omega_1\}$. Define a transfinite sequence $\{s_{\iota} \mid \iota \in \omega_1\}$ by induction: We set $s_0 = 0$ and $s_{\iota} = \min \{\{\omega_1 - \{\alpha \in \omega_1$ there is a β, $\beta < \iota$ such that $\rho(d_n^{\alpha}, d_m^{s_{\beta}}) < \frac{\varepsilon}{3}$ for some $n,m \in \omega\}\}$. Put $P_{\iota} = \{d_n^{s_{\iota}} \mid n \in \omega\}$. The family $\{P_{\iota} \mid \iota \in \omega_1\}$ is $\frac{\varepsilon}{3}$-discrete but it is not σ-discrete in X.

The third example we want to show concerns a conceptual question of the nonseparable descriptive theory of sets. Following the definition of Z. Frolík and P. Holický [FH$_1$], a uniform space X is called analytic if there exists an upper semicontinuous compact-space-valued correspondence $f: M \rightarrow X$ from a complete metric space M onto X such that the images of σ-d.d. families in M are σ-d.d. families in X. A natural question appears if it is possible to replace the words " σ-d.d. families" in the latter definition by the words "σ-discrete families", obviously without changing the meaning of analycity. The following example shows that it is not.

Example 2.3: There exists a σ-d.d. in itself space X with the property: If $f:M \to X$ is an upper semicontinuous compact-space-valued correspondence of a complete metric space onto X then there is a discrete family $\{M_\alpha \mid \alpha \in I\}$ in M such that $\{f(M_\alpha) \mid \alpha \in I\}$ is not σ-discrete.

Let S be a set, Card S = ω_1. Put $X = S \times \omega_0$ and endow X with the uniformity which has the coverings of the type $\mathcal{X}_F = \{\{(s,\alpha)\} \mid s \in F,\ F \text{ finite},\ \alpha \in \omega_0\} \cup \{t \times \omega_0 \mid t \in S - F\}$ for a base. Of course, X is σ-d.d. in itself and therefore analytic.

Now, suppose that there is an upper semicontinuous compact-space-valued correspondence $f:M \to X$ from a complete metric M onto X. Take the family $\{f^{-1}(s,0) \mid s \in S\}$. This family is point finite and completely additive family of closed sets in M and therefore it must be σ-d.d. (see [KP]). As a consequence, we may find a set S', $S' \subset S$, card $S' = \omega_1$ such that, for some points $x_s \in f(s,0)$, $s \in S'$, the family $\{x_s \mid s \in S'\}$ is discrete in M. As any $f(x_s)$ must be finite (being compact in X) and the family $\{f(x_s), s \in S'\}$ point countable, we can find a set S'', $S'' \subset S'$, card $S'' = \omega_1$ such that $\{f(x_s) \mid s \in S''\}$ consists of pairwise disjoint sets. Further, observe that $\{x_s \mid s \in S''\}$ and $f^{-1}(X - \bigcup_{s \in S''} f(x_s))$ are closed disjoint sets in M and therefore any x_s is far from $f^{-1}(X - \bigcup_{s \in S''} f(x_s))$ within some $\frac{1}{n}$, $n \in N$. Hence there is a set S''', $S''' \subset S''$, card $S''' = \omega_1$ such that the sets $\{x_s \mid s \in S'''\}$ and $f^{-1}(X - \bigcup_{s \in S''} f(x_s))$ are uniformly far. Since any $f(x_s)$ is finite, we can take the set S''' such that $(\{s'\} \times \omega_0) \cap f(x_s) = \emptyset$ for any different $s, s' \in S'''$.

Now, consider the family $\{f(x_s) \mid s \in S'''\}$. Any $f(x_s)$ is finite and so there is some $\alpha_s \in \omega_0$ that $(s, \alpha_s) \notin f(x_s)$.

By a similar argument as we have presented formerly, we can find a set S^{IV}, $S^{IV} \subset S^{III}$, card $S^{IV} = \omega_1$ and points $y_s, y_s \in M$, $s \in S^{IV}$ such that the family $\{y_s \mid s \in S^{IV}\}$ is discrete, $\{f(y_s) \mid s \in S^{IV}\}$ disjoint and $(s, \alpha_s) \in f(y_s)$ for any $s \in S^{IV}$. Since the set $P = \{x_s \mid s \in S^{IV}\} \cup \{y_s \mid s \in S^{IV}\}$ is discrete (and so any disjoint family of subsets of P is discrete, too) and since the correspondence f is supposed to map discrete collection on σ-discrete ones, we conclude that any disjoint family in the space $Q = \{(s,0) \mid s \in S^{IV}\} \cup \{(s, \alpha_s) \mid s \in S^{IV}\}$ is σ-discrete. We are approaching a contradiction. First, Q is uniformly isomorphic to the space $S^{IV} \times \{0,1\}$ endowed with the uniformity which has for a base coverings of the type $\mathcal{X}_F = \{\{(s,\alpha)\} \mid s \in F, F \text{ finite}, \alpha = 0,1\} \cup \{\{s\} \times \{0,1\} \mid s \in S^{IV} - F\}$. We are to show that there is a disjoint family of subsets of $S^{IV} \times \{0,1\}$ which is not σ-discrete. Take a partition of S^{IV} consisting of ω_1 classes, each of cardinality ω_1. So we have $S^{IV} = \bigvee_{\beta \in \omega_1} S_\beta$, card $S_\beta = \omega_1$. Choose a well ordering $<_\beta$ on S_β and denote by u_β^γ the γ-th element of S_β. Set $R_\gamma = (S_\gamma \times \{0\}) \cup (\bigcup_{\beta \in \omega_1} \{u_\beta^\gamma\} \times 1)$. The family $\{R_\gamma \mid \gamma \in \omega_1\}$ is not σ-discrete because only finite subfamilies are discrete. The proof is finished.

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