

Jaroslav Ježek; Tomáš Kepka

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SEMIGROUP REPRESENTATIONS OF MEDIAL GROUPOIDS

Jaroslav JEŽEK, Tomáš KEPKA

Abstract: For every medial groupoid G with $GG = G$, there exist a commutative monoid $S(+,0)$ and its two commuting automorphisms f and g such that $G \cong S$ and $xy = f(x) + g(y)$ for all $x, y \in G$.

Key words: Semigroup, representation, medial groupoid.

Classification: 08A05

In the past, a considerable attention was paid to the problem of representations of medial groupoids by means of commutative semigroups and their commuting endomorphisms (see e.g. [1],[2],[3],[6],[7],[8],[9] and [10]). In the present paper, we are going to show that all medial groupoids without irreducible elements and all commutative medial groupoids have semigroup representations.

1. Preliminaries. Throughout this paper, let \mathbb{E} be a free monoid over a two-element set $\{\alpha, \beta\}$. Every element e of \mathbb{E} can be written uniquely as $e = a_1 \dots a_n$ for some $n \geq 0$ and $a_1, \dots, a_n \in \{\alpha, \beta\}$. We put $d(e) = n$ and $\mathbb{E}_m = \{f \in \mathbb{E}; d(f) = m\}$ for all $m \geq 0$. Further, $l(e) = \text{card} \{i; a_i = \alpha\}$, $r(e) = \text{card} \{i; a_i = \beta\}$ and $\mathbb{E}_{i,j} = \{e \in \mathbb{E}; (l(e), r(e)) = (i, j)\}$.

Let X be a non-empty set and SW'_X the free algebra over X in the variety of universal algebras of the type $\{+, 0, \alpha, \beta\}$ satisfying the identities $(x + y) + z = x + (y + z)$, $x + y = y + x$, $x + 0 = x$, $\alpha(x + y) = \alpha x + \alpha y$, $\beta(x + y) = \beta x + \beta y$, $\alpha 0 = 0$, $\beta 0 = 0$. Elements from SW'_X are called semiterms over X . Every semiterm can be expressed uniquely (up to the order of summands) in the form $s = e_1 x_1 + \dots + e_n x_n$, where $n \geq 0$, $e_i \in E$ and $x_i \in X$. We put $d(s) = \max d(e_i)$, $I^*(s) = \{e_1, \dots, e_n\}$, $I(s) = \{e \in E; e f \in I^*(s) \text{ for some } f \in E\}$.

Define a binary operation (denoted multiplicatively) on SW'_X by $rs = \alpha r + \beta s$. We obtain a groupoid SW . The subgroupoid W_X generated by X is an absolutely free groupoid over X and its elements are called terms.

Let $r \in W$ be a term and $e \in I(r)$. Then there exists a unique pair u, v such that u is a semiterm, v is a term and $r = u + ev$. We put $v = r_{[e]}$.

2. Linear representations of medial groupoids. Let X be a non-empty set. Denote by F'_X the free algebra over X in the variety \mathcal{R} of universal algebras of the type $\{+, 0, \alpha, \beta\}$ satisfying the identities $(x + y) + z = x + (y + z)$, $x + y = y + x$, $x + 0 = x$, $\alpha(x + y) = \alpha x + \alpha y$, $\beta(x + y) = \beta x + \beta y$, $\alpha 0 = 0$, $\beta 0 = 0$, $\beta \alpha x = \alpha \beta x$. Every element of F'_X can be written in the form $u = \sum_{i=1}^k \alpha^{n_i} \beta^{m_i} x_i$, where r, n_i, m_i are non-negative integers and $x_i \in X$; this expression is unique up to the order of summands. We put $d(u) = \max (n_i + m_i)$.

Define a multiplication on F'_X by $uv = \alpha u + \beta v$. The set F'_X together with this operation is a groupoid which will

be denoted by F_X . Moreover, we can identify the set F_X with a subset of SW_X . It is easy to see that the groupoid F_X is a medial cancellation groupoid, and so it is entropic (recall that a groupoid is said to be medial if it satisfies the identity $xy.uv = xu.yv$ and it is said to be entropic if it is a homomorphic image of a medial cancellation groupoid). Denote by G_X the subgroupoid of F_X generated by X . According to [4, Theorem 2.1], G_X is a free entropic groupoid over X .

Let G be a groupoid. By a linear representation of G we mean an algebra $S(+,0,f,g) \in \mathcal{R}$ (i.e., $S(+,0)$ is a commutative monoid and f,g are commuting 0-preserving endomorphisms of $S(+)$) together with an element $e \in S$ such that G is a subset of S and $xy = f(x) + g(y) + e$ for all $x,y \in G$. The representation is called exact if $S = G$ and it is called convex if $e = 0$.

Using the fact that the underlying semigroups of free \mathcal{R} -algebras are cancellative, it is easy to show that groupoids with linear representations are entropic. Conversely, every medial groupoid containing an element a such that the corresponding translations are permutations has an exact linear representation (see [9]). Further, every regular medial division groupoid has an exact linear representation (see [6]) and every medial cancellation groupoid has a convex linear representation.

3. Representations of medial groupoids without irreducible elements. The purpose of this section is to prove the following

Theorem 1. Let G be a medial groupoid such that $GG = G$. Then G has a convex linear representation $S(+, 0, f, g)$ such that f, g are automorphisms of $S(+)$.

The proof of this result will be divided into several lemmas.

Let $A(\circ)$ be a medial groupoid such that $A = A \circ A$. By [5, Proposition 4.3], $A(\circ)$ is entropic. In the following, we shall use the groupoids F_A and G_A defined in the preceding section. Denote by h the homomorphism of G_A onto $A(\circ)$ such that $h(x) = x$ for every $x \in A$. Further, for every $x \in A$, fix elements $p_\alpha(x), p_\beta(x)$ in A such that $x = p_\alpha(x) \circ p_\beta(x)$. For every $x \in A$ and every $e \in E$, we define an element $p_e(x)$ of A by induction on $d(e)$ as follows: $p_1(x) = x$; $p_{e\alpha}(x) = p_\alpha(p_e(x))$; $p_{e\beta}(x) = p_\beta(p_e(x))$. Finally, for every $x \in A$ and every non-negative integer n , denote by $C(n, x)$ the element $\sum_{e \in E_n} \alpha^{l(e)} \beta^{r(e)} p_e(x)$ of F_A .

Lemma 2. Let $x \in A$ and $n \geq 0$. Then $C(n, x) \in G_A$ and $h(C(n, x)) = x$.

Proof. We shall proceed by induction on n . Let $n \geq 1$. Then $C(n, x) = C(n-1, p_\alpha(x)) \cdot C(n-1, p_\beta(x))$, and so $h(C(n, x)) = p_\alpha(x) \circ p_\beta(x) = x$.

For every element $u = \sum_{i=1}^k \alpha^{n_i} \beta^{m_i} x_i$ of H_A (see [4]) and every integer $n \geq d(u)$, let $D(n, u) = \sum_{i=1}^k \alpha^{n_i} \beta^{m_i} C(n-n_i-m_i, x_i)$.

Lemma 3. Let $u = \sum_{i=1}^k \alpha^{n_i} \beta^{m_i} x_i \in H_A$. Then:

(i) $D(n, u) \in G_A$ for every $n \geq d(u)$.

(ii) $D(n,u) = \sum_{i=1}^s \sum_{e \in E_{n-n_i-m_i}} \alpha^{n_i+1(e)} \beta^{m_i+r(e)} p_e(x_i)$. Moreover, $n_i + 1(e) + m_i + r(e) = n$ for all $1 \leq i \leq s$ and $e \in E_{n-n_i-m_i}$.

Proof. For every $0 \leq k \leq n$ we have $c(n,k,D(n,u)) = \sum_{i=1}^s \sum_{e \in E_{n-n_i-m_i}} \binom{0}{k-n_i-1(e)} = \sum_{i=1}^s \text{card} \{ e \in E_{n-n_i-m_i}; k = n_i + 1(e) \} = \sum_{i=1}^s \binom{n-n_i-m_i}{k-n_i} = c(n,k,u) = \binom{n}{k}$, since $u \in H_A$ (see [4]). We have proved that $D(n,u) \in H_A$. By [4, Lemmas 2.10, 2.11], $D(n,u) \in G_A$ and the rest is clear.

Define a binary relation R on F_A as follows: $(u,v) \in R$ iff there are $w \in F_A$, $x,y,z \in A$ and $c,d \geq 0$ such that $u = w + \alpha^c \beta^d x$, $v = w + \alpha^{c+1} \beta^d y + \alpha^c \beta^{d+1} z$ and $x = y \circ z$. Further, define a binary relation S on F_A by $(u,v) \in S$ iff there are $m \geq 0$ and $u_0, \dots, u_m \in F_A$ such that $u = u_0$, $v = u_m$ and $(u_{i-1}, u_i) \in R \cup R^{-1}$ for all $1 \leq i \leq m$. Evidently, S is a congruence of the algebra $F'_A = F_A(+, 0, \alpha, \beta)$.

Lemma 4. Let $(u,v) \in S$. Then $u \in H_A$ iff $v \in H_A$.

Proof. See [4, Lemma 2.9].

Lemma 5. Let $u,v \in H_A$ and $(u,v) \in R$. Then $h(D(n,u)) = h(D(n,v))$ for every $n \geq d(v)$.

Proof. Let $n \geq d(v)$. We have $u = w + \alpha^c \beta^d x$ and $v = w + \alpha^{c+1} \beta^d y + \alpha^c \beta^{d+1} z$ for some $w \in F_A$, $c,d \geq 0$ and $x,y,z \in A$ with $x = y \circ z$. There is $w' \in F_A$ such that $D(n,u) = w' + \alpha^c \beta^d C(n-c-d,x)$ and $D(n,v) = w' + \alpha^{c+1} \beta^d C(n-c-d-1,y) + \alpha^c \beta^{d+1} C(n-c-d-1,z)$. We can express w' in the form $w' = \sum_{i=1}^s \alpha^{n_i} \beta^{m_i} x_i$; by Lemma 3, $D(n,u) \in H_A$ and $n_i + m_i = n$

for all $i = 1, \dots, s$. Hence, for every $0 \leq k \leq n$,

$$\binom{n}{k} = c(n, k, w') + \sum_{e \in E_{n-c-d}} \binom{n-c-d-l(e)-r(e)}{k-c-l(e)} = \text{card} \{i; n_i = k\} + \text{card} \{e \in E_{n-c-d}; l(e) = k - c\} = \text{card} \{i; n_i = k\} + \binom{n-c-d}{k-c},$$

so that $\text{card} \{i; n_i = k\} = \binom{n}{k} - \binom{n-c-d}{k-c}$. Now, let P designate the set of all $a_1, \dots, a_n \in E_n$ such that $a_1 = \dots = a_c = \alpha$ and $a_{c+1} = \dots = a_{c+d} = \beta$. We have

$$\text{card} (E_{k, n-k} \setminus P) = \binom{n}{k} - \binom{n-c-d}{k-c}$$

for every $0 \leq k \leq n$. There exists a bijective mapping t_k of $E_{k, n-k} \setminus P$ onto $\{i; n_i = k\}$.

Using this, let us define a mapping t of E_n into A as follows: $t(e) = x_{t_k(e)}$ for each $e \in E_{k, n-k} \setminus P$; $t(e) =$

$$p_{a_{c+d+1} \dots a_n}(x) \text{ for every } e = a_1 \dots a_n \in P. \text{ It is now easy to check that } D(n, u) = \sum_{e \in E_n} \alpha^{l(e)} \beta^{r(e)} t(e).$$

There exists a unique term $r \in W_A$ with $I(r) = E_0 \cup \dots \cup E_n$ and $r_{[e]} = t(e)$ for every $e \in E_n$. Denote by g the homomorphism of W_A onto G_A such that $g(x) = x$ for all $x \in A$. We have

$g(r) = D(n, u)$ by [4, Lemma 2.2]. Further, denote by f the element a_1, \dots, a_{c+d} of E_{c+d} with $a_1 = \dots = a_c = \alpha$ and $a_{c+1} = \dots = a_{c+d} = \beta$. Clearly, $g(r_{[f]}) = C(n-c-d, x)$. Moreover,

$$\begin{aligned} & \text{there are elements } u_1, \dots, u_d, v_1, \dots, v_c \in W_A \text{ such that } r = \\ & = (((u_d(\dots(u_2(u_1, r_{[f]}))))v_1)v_2)\dots)v_c. \text{ Consequently,} \\ & D(n, u) = g(r) = (((u'_d(\dots(u'_2(u'_1, C(n-c-d, x))))))v'_1)v'_2)\dots)v'_c \\ & \text{where } u'_i = g(u_i) \text{ and } v'_i = g(v_i) \text{ are elements from } G_A. \text{ Put} \\ & q = (((u'_d(\dots(u'_2(u'_1, C(n-c-d-1, y)C(n-c-d-1, z))))))v'_1)\dots)v'_c. \\ & \text{Then } D(n, u) = \alpha^c \beta^d C(n-c-d, x) + \sum_{k=1}^d \alpha^{c+1} \beta^{d-k} u'_k + \\ & + \sum_{j=1}^c \alpha^{c-1} \beta v'_j \text{ and so } w' = \sum_{k=1}^d \alpha^{c+1} \beta^{d-k} u'_k + \sum_{j=1}^c \alpha^{c-1} \beta v'_j. \end{aligned}$$

Further, $q = \alpha^c \beta^d (C(n-c-d-1, y) \cdot C(n-c-d-1, z)) +$
 $+ \sum_{k=1}^d \alpha^{c+1} \beta^{d-k} u'_k + \sum_{j=1}^c \alpha^{c-1} \beta^j v'_j = \alpha^c \beta^d (C(n-c-d-1, y) \cdot$
 $\cdot C(n-c-d-1, z)) + w' = \alpha^{c+1} \beta^d C(n-c-d-1, y) +$
 $+ \alpha^c \beta^{d+1} C(n-c-d-1, z) + w' = D(n, v)$. Therefore $h(D(n, u)) =$
 $= (((h(u'_d) \circ \dots \circ (h(u'_1) \circ h(C(n-c-d, x)))) \circ h(v'_1)) \circ h(v'_2)) \circ$
 $\circ \dots \circ h(v'_c)$ and $h(D(n, v)) =$
 $= (((h(u'_d) \circ \dots \circ (h(u'_1) \circ h(C(n-c-d-1, y) \cdot C(n-c-d-1, z)))) \circ$
 $\circ h(v'_1) \circ \dots \circ h(v'_c)$. But $h(C(n-c-d, x)) = x$, $h(C(n-c-d-1, y)) =$
 $= y$ and $h(C(n-c-d-1, z)) = z$ by Lemma 2. Finally, $x = y \circ z$ and
we see that $h(D(n, u)) = h(D(n, v))$.

Lemma 6. Let $x, y \in A$ be such that $(x, y) \in S$. Then $x = y$.

Proof. There are elements $u_0, \dots, u_m \in F_A$ such that $x =$
 $= u_0$, $y = u_m$ and $(u_{i-1}, u_i) \in R \cup R^{-1}$. By Lemma 4, $u_i \in H_A$. Let n
be such that $n \geq d(u_i)$ for all i . By 3.5, $h(D(n, u_0)) = \dots =$
 $= h(D(n, u_m))$. Thus $x = h(D(n, u_0)) = h(D(n, u_m)) = y$.

Lemma 7. Let $u, v \in F_A$ be such that either $(\alpha u, \alpha v) \in S$
or $(\beta u, \beta v) \in S$. Then $(u, v) \in S$.

Proof. It is easy to see that if $(p, q) \in R \cup R^{-1}$ and $p =$
 $= \alpha r$ for some r then $q = \alpha s$ for some s and $(r, s) \in R \cup R^{-1}$;
similarly for β .

Lemma 8. The groupoid $A(\circ)$ has a convex linear repre-
sentation such that f, g are injective.

Proof. It follows from the definition of S and from
Lemma 6 that an algebra isomorphic to $F_A(+, 0, \alpha, \beta)/S$ is a
convex linear representation of $A(\circ)$. Let $S(+, 0, f, g)$ be such
an algebra. By Lemma 7, both f and g are injective and pre-
serve the element 0.

Now, Theorem 1 is an easy consequence of Lemma 8.

4. Representations of medial groupoids with zero and without irreducible elements

Proposition 9. Let G be a medial groupoid such that $GG = G$. Suppose that G contains a zero element o (i.e., $xo = o = ox$). Then G has a convex linear representation $S(+, 0, f, g)$ such that f, g are automorphisms of $S(+)$ and $x + o = o$ for all $x \in S$.

The proof of this result will be divided into six lemmas. Let $A(o)$ be a medial groupoid with $A \circ A = A$ and let o be a zero element of $A(o)$. We keep the notation of the preceding section; in the present case, we can assume that $p_e(o) = o$ for every $e \in E$.

Lemma 10. Let $u = \sum_{i=1}^s \alpha^{n_i} \beta^{m_i} x_i \in H_A$ be such that $x_i = o$ for some $1 \leq i \leq s$. Then $h(D(n, u)) = o$ for every $n \geq d(u)$.

Proof. $D(n, u) = \sum_{i=1}^s \alpha^{n_i} \beta^{m_i} C(n - n_i - m_i, x_i) = \sum_{i=1}^{2^m} \alpha^{c_i} \beta^{d_i} y_i$ for some $c_i, d_i \geq 0$ and $y_i \in A$ such that $c_i + d_i = n$ and o appears among the elements y_i . Further, there is a $t \in W_A$ such that $I(t) = E_0 \cup \dots \cup E_n$, o is contained in t and $h(t) = D(n, u)$. Denote by g the homomorphism of W_A onto G_A such that $g(x) = x$ for all $x \in A$. We have $h(D(n, u)) = hg(t) = o$.

Let I be the set of all $u = \sum_{i=1}^s \alpha^{n_i} \beta^{m_i} x_i \in F_A$ such that $x_i = o$ for some i and define a binary relation Q on F_A as follows: $(u, v) \in Q$ iff either $(u, v) \in S$ or there exist elements $w, z \in I$ with $(u, w) \in S$ and $(v, z) \in S$.

Lemma 11. I is an ideal of $F_A(+)$ and Q is a congruence of $F_A(+, 0, \alpha, \beta)$.

Proof. Obvious.

Lemma 12. Let $x \in A$, $x \neq 0$, $u \in F_A$ and $(x, u) \in S$. Then $u \notin I$.

Proof. There are $m \geq 0$ and elements $u_0, \dots, u_m \in F_A$ such that $x = u_0$, $u = u_m$ and $(u_{i-1}, u_i) \in R \cup R^{-1}$. Moreover, there is a positive integer n such that $n \geq d(u_i)$ for all i . Since $x \in H_A$, we have $u_i \in H_A$ and $h(D(n, u_{i-1})) = h(D(n, u_i))$ by Lemma 5. However, $h(D(n, u_0)) = x$, and hence $h(D(n, u)) = x$. By Lemma 10, $u \notin I$.

Lemma 13. Let $x, y \in A$ be such that $(x, y) \in Q$. Then $x = y$.

Proof. Suppose $x \neq y$. Then at least one of these elements is different from 0; by Lemma 12 and the definition of Q , we get $(x, y) \in S$. Now, $x = y$ by Lemma 7, a contradiction.

Lemma 14. Let $u, v \in F_A$ be such that either $(\alpha u, \alpha v) \in Q$ or $(\beta u, \beta v) \in Q$. Then $(u, v) \in Q$.

Proof. Easy.

Lemma 15. The groupoid $A(0)$ has a convex linear representation $S(+, 0, f, g)$ such that f, g are injective, $x + 0 = 0$ for all $x \in S$ and $f(0) = 0 = g(0)$.

Proof. An algebra isomorphic to $F_A(+, 0, \alpha, \beta)/Q$ has the required properties.

Now, Proposition 9 is an easy consequence of Lemma 15.

5. Linear representations of commutative medial groupoids

Theorem 16. Let G be a commutative medial groupoid. Then G has a convex linear representation $S(+, 0, f, g)$ such that

$f = g$ and f is an automorphism of $S(+)$.

The proof of the result will be divided into four lemmas.

Let $A(o)$ be a commutative medial groupoid. We denote by h the unique homomorphism of CG_A onto $A(o)$ such that $h(x) = x$ for each $x \in A$ (see [4, section 3]).

Define a binary relation R on CF_A as follows: $(u,v) \in R$ iff there are $w \in CF_A$, $x,y,z \in A$ and $c \geq 0$ such that $u = w + \alpha^c x$, $v = w + \alpha^{c+1} y + \alpha^{c+1} z$ and $x = y \circ z$. Further, define a binary relation S on CF_A by $(u,v) \in S$ iff there are $m \geq 0$ and $u_0, \dots, u_m \in CF_A$ such that $u_0 = u$, $u_m = v$ and $(u_{i-1}, u_i) \in R \cup R^{-1}$.

Lemma 17. (i) S is a congruence of $CF_A(+, 0, \alpha)$.

(ii) If $u, v \in CF_A$ and $(\alpha u, \alpha v) \in S$ then $(u, v) \in S$.

(iii) If $(u, v) \in S$ then $u \in CG_A$ iff $v \in CG_A$.

(iv) If $x, y, z \in A$ and $x = y \circ z$ then $(x, yz) \in S$.

Proof. Easy (see [4, Lemma 3.4]).

Lemma 18. Let $(u, v) \in R$ and $u, v \in CG_A$. Then $h(u) = h(v)$.

Proof. We have $u = w + \alpha^c x$ and $v = w + \alpha^{c+1} y + \alpha^{c+1} z$, $w \in CF_A$, $c \geq 0$, $x, y, z \in A$, $x = y \circ z$. Denote by g the homomorphism of W_A onto G_A with $g(x) = x$ for every $x \in A$. Then $g(t) = u$ for some $t \in W_A$. By [4, Lemma 3.2], there is an $e = a_1 \dots a_c \in I(t)$ such that $t_{[e]} = x$. It is easy to see that $t = [t_{[e]}, a_c, t_c, \dots, a_1, t_1]$ for some $t_1, \dots, t_c \in W_A$ (see [5, Proposition 1.7]). Put $u_1 = g(t_1), \dots, u_c = g(t_c)$. Then $u_1, \dots, u_c \in CG_A$ and $u = g(t) = ((xu_c)u_{c-1}) \dots u_1$. Consequently, $u = \alpha^c x + \alpha^c u_c + \alpha^{c-1} u_{c-1} + \dots + \alpha u_1$, $v = \alpha^{c+1} y + \alpha^{c+1} z + \alpha^c u_c + \alpha^{c-1} u_{c-1} + \dots + \alpha u_1 = ((yz \cdot u_c)u_{c-1}) \dots u_1$. From this, $h(u) = h(v)$.

Lemma 19. Let $x, y \in A$ and $(x, y) \in S$. Then $x = y$.

Proof. Use Lemma 17(ii) and Lemma 18.

Lemma 20. The groupoid $A(o)$ has a convex linear representation $S(+, 0, f, f)$ such that f is injective.

Proof. Use Lemma 17 and Lemma 19.

Now, Theorem 16 is an easy consequence of Lemma 20.

6. Remarks

Proposition 21. Let f and g be commuting endomorphisms of a commutative semigroup $S(+)$ and $e \in S$. Put $xy = f(x) + g(y) + e$ for all $x, y \in S$ and suppose that the medial groupoid S is divisible. Then it is regular.

Proof. Let $L_a(b) = a + b$ for all $a, b \in S$. Denote by T the set of all $a \in S$ such that L_a is a projective transformation of S . Since the groupoid S is divisible, $e \in T$ and $f(a), g(a) \in T$ for each $a \in S$. On the other hand, T is a subsemigroup and, moreover, an abelian group. The rest is clear.

Example 22. Let G be a non-regular medial division groupoid (see [7]). Then G has a convex linear representation. According to Proposition 21, G has no exact linear representation.

Example 23. Let $X = \{x, y\}$ be a two-element set. Denote by L the twelve-element subset of G_X formed by the elements $x, xx, xx.x, x.xx, xx.xx, (xx)(xx.x), y, xy, xy.x, (xy.x)(xx), (xy.x)(x.xx), ((xy.x)(xx))((xx)(xx.x))$. It is easy to see that $J = G_X \setminus L$ is an ideal of G_X . Put $r = (J \times J) \cup \text{id}$ and $A = G_X/r$; it is clear that r is a congruence. We obtain thus an entropic

groupoid A . On the other hand, it is easy to check that A has no convex linear representation.

Remark 24. The following problem seems to be open: Has every entropic groupoid a linear representation?

R e f e r e n c e s

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Matematicko-fyzikální fakulta, Universita Karlova,
Sokolovská 83, 18600 Praha 8, Československo

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