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DISCONNECTED REGULAR s-MANIFOLDS

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Abstract: The author presents some typical constructions of disconnected regular s-manifolds i.e. of certain distributive groupoids on smooth manifolds which generalize the notion of a symmetric space in two directions: The symmetries are not necessarily involutive and the space may have more than one component.

Key words: Generalized symmetric spaces, regular s-manifolds, distributive groupoids.

Classification: 53C35

Introduction. Following O. Kowalski [1],[2], a regular s-manifold is a manifold M with a differentiable multiplication $\mu: M \times M \longrightarrow M$ written as $\mu(x,y) = x \cdot y$ such that the maps $s_x: M \longrightarrow M$, $x \in M$, given by $s_x(y) = x \cdot y$ satisfy the following axioms:

(i) $s_x(x) = x$,

(ii) each s_x is a diffeomorphism,

(iii) $s_x \circ s_y = s_z \circ s_x$, where $z = s_x(y)$,

(iv) for each $x \in M$, the tangent map $(s_x)_{*x}: T_x(M) \longrightarrow T_x(M)$ has no fixed vectors except the null vector.

The diffeomorphism s_x, $x \in M$ are called symmetries of M:

An automorphism of (M, μ) is a diffeomorphism $\phi: M \longrightarrow M$ such that $\phi(x \cdot y) = \phi(x) \cdot \phi(y)$ for every $x.y \in M$. Obviously, all symmetries s_x of (M, μ) are automorphisms due to axioms (ii) and (iii).

In the definition of a regular s-manifold one does not suppose that the underlying manifold M is connected. Yet the book [1] is devoted, in fact, to the theory of connected regular s-manifolds.

The disconnected regular s-manifolds apparently require a special theory, which may be non-trivial (see the examples 1)-4) in [1], p. 66). Here we develop some more basic facts and constructions concerning disconnected regular s-manifolds. At the same time, we generalize the examples mentioned above.

§ 1. Let $(M_\alpha, \{s_x^\alpha\})$, $\alpha \in A$, be a set of connected regular s-manifolds. Let $M = \bigvee_{\alpha \in A} M_\alpha$ be the disjoint sum of the underlying manifolds.

Definition 1. A regular s-manifold $(M, \{s_x\})$ will be said to be composed of the $(M_\alpha, \{s_{x_\alpha}^\alpha\})$ if for every $\alpha \in A$, $x_\alpha \in M_\alpha$ we have

$$(1) \qquad s_{x_\alpha} | M_\alpha = s_{x_\alpha}^\alpha .$$

It is obvious that every disconnected regular s-manifold is composed of its connected components in the above sense. Here the regular s-structures on the connected components are determined by (1).

Proposition 1. If $(M, \{s_x\})$ is a regular s-manifold which is composed of the connected regular s-manifolds $(M_\alpha, \{s_{x_\alpha}^\alpha\})$, $\alpha \in A$, then the index set A has a natural structure of a 0-dimensional regular s-manifold.

<u>Proof</u>. For any two $\alpha, \beta \in A$ consider the maps $s_{x_\alpha}|M_\beta$, where x_α runs over M_α.
Because each $s_{x_\alpha}: M \longrightarrow M$ is a diffeomorphism, it maps each connected component onto a connected component. Because the map $(x_\alpha, x_\beta) \mapsto s_{x_\alpha}(x_\beta)$ is smooth for a given $x_\beta \in M_\beta$, we see that the connected component $s_{x_\alpha}(M_\beta) = M_\gamma$ does not depend on the choice of $x_\alpha \in M_\alpha$.
Thus, we have a uniquely determined index $\gamma = \alpha \cdot \beta$. It is clear that $\alpha \cdot \alpha = \alpha$ and, for each $\alpha \in A$, the map $L_\alpha : \beta \longrightarrow \alpha \cdot \beta$ is one-to-one on A.

Finally, consider the regularity condition

$$s_{x_\alpha} \circ s_{x_\beta} = s_{s_{x_\alpha}(x_\beta)} \circ s_{x_\alpha} \text{ on } M_\gamma .$$

We obtain

$$\alpha \cdot (\beta \cdot \gamma) = (\alpha \cdot \beta) \cdot (\alpha \cdot \gamma)$$

which is the regularity condition for A.

Hence A with the multiplication $(\alpha, \beta) \longrightarrow \alpha \cdot \beta$ is a 0-dimensional regular s-manifold.

Q.E.D.

<u>Definition 2</u>. The regular s-manifold $(A, \cdot)$ will be called the <u>index groupoid</u> of $(M, \{s_x\})$.

<u>Proposition 2</u>. Let $(M, \{s_x\})$ be composed of $(M_\alpha, \{s^\alpha_{x_\alpha}\})$, $\alpha \in A$, in such a way that the index groupoid $(A, \cdot)$ is transitive (i.e., the transformation group G generated by all maps L_α, $\alpha \in A$, is transitive on A).
Then all components $(M_\alpha, \{s^\alpha_{x_\alpha}\})$, $\alpha \in A$ are isomorphic to the same (connected) regular s-manifold $(M_o, \{s^o_u\})$.

<u>Proof</u>. It is sufficient to prove the following:

if $\gamma \cdot \alpha = \beta$ for 3 indices $\alpha, \beta, \gamma \in A$, then $(M_\alpha, \{s^\alpha_{x_\alpha}\})$ is isomorphic to $(M_\beta, \{s^\beta_{x_\beta}\})$. But for any element $x_\gamma \in M_\gamma$, $s_{x_\gamma}|M_\alpha$ is a diffeomorphism of M_α onto M_β.

Further, from the regularity of $(M, \{s_x\})$ we get

$$s_{x_\gamma} \circ s_{x_\alpha}|M_\alpha = s_{s_{x_\gamma}(x_\alpha)} \circ s_{x_\gamma}|M_\alpha ,$$

where $s_{x_\gamma}(x_\alpha) = x_\beta \in M_\beta$.

But this is just the isomorphism between $(M_\alpha, \{s^\alpha_{x_\alpha}\})$ and $(M_\beta, \{s^\beta_{x_\beta}\})$. The structure of "transitively composed" regular s-manifolds is not easy to describe. Yet, we shall show the construction of a special class, where we do not suppose the transitivity of the index groupoid but only the isomorphism of the components.

Proposition 3. Let $(A, \cdot)$ be a 0-dimensional regular s-manifold and $(M_0, \{s^0_u\})$ a "model" regular s-manifold. Then the direct product $(A, \cdot) \times (M_0, \{s^0_u\})$ is a regular s-manifold with the index groupoid $(A, \cdot)$.

Proof is obvious. We only recall that the composed manifold $(A \times M_0, \{s_w\})$ is defined by the formula

$$(2) \qquad s_{(\alpha,u)}(\beta, v) = (\alpha \cdot \beta,\ s^0_u(v)), \qquad u, v \in M_0,\ \alpha, \beta \in A$$

Now, examples 2, 3 from [1] are special cases of Proposition 3.

If the groupoid A is trivial in the sense that $\alpha \cdot \beta = \beta$ for any $\alpha, \beta \in A$, then we see easily that

$$s_{(\alpha,u)}(\beta, v) = (\beta, s_u(v)) \text{ for any } u, v \in M_0$$

and this is Example 2.

Example 3 is obtained for a groupoid A consisting of 3 elements (1,2,3) with a transitive multiplication.

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We shall now generalize example 1 from [1].
Let us consider again a 0-dimensional regular s-manifold $(A,\cdot)$ and the group G generated by all the left translations L_{α}: $\beta \to \alpha\cdot\beta$, $\beta \in A$. G is a group of automorphisms of the groupoid $(A,\cdot)$. Let us consider a relation $\cong$ on $(A,\cdot)$ defined as follows: $\alpha \cong \beta$ if and only if α belongs to the orbit of β with respect to the group G, i.e. if and only if $\alpha = g(\beta)$ for some $g \in G$.

In particular, our relation is an equivalence relation, and the following is satisfied:

a) $\beta \cong \gamma \Longleftrightarrow \alpha\cdot\beta \cong \alpha\cdot\gamma$,

b) $\alpha \cong \beta\cdot\gamma \Longleftrightarrow \alpha \cong \gamma$.

<u>Proposition 4</u>. Let $(M, \{s_x\})$ be composed of $(M_\alpha, \{s^{\alpha}_{x_\alpha}\})$, $\alpha \in A$, with the index groupoid $(A,\cdot)$. For every $\alpha, \beta \in A$, the relation $\alpha \cong \beta$ implies the isomorphism between $(M_\alpha, \{s^{\alpha}_{x_\alpha}\})$ and $(M_\beta, \{s^{\beta}_{x_\beta}\})$.

<u>Proof</u> is the same as for Proposition 2.

<u>Proposition 5</u>. Let $(A,\cdot)$ be a 0-dimensional regular s-manifold with the corresponding equivalence relation $\cong$. Let $(M_\alpha, \{s^{\alpha}_{x_\alpha}\})_{\alpha\in A}$ be a family of connected regular s-manifolds such that, for every two indices $\alpha \cong \beta$, the regular s-manifolds $(M_\alpha, \{s^{\alpha}_{x_\alpha}\})$, $(M_\beta, \{s^{\beta}_{x_\beta}\})$ are isomorphic to the same regular s-manifold $(M_{[\alpha]}, \{s^{[\alpha]}_u\})$, where $[\alpha]$ means the equivalence class of α in A.

Put $M = \{(\alpha,u) \mid \alpha \in A,\ u \in M_{[\alpha]}\}$ and, for each $(\alpha,u) \in M$

define the transformations $s_{(\alpha,u)}$ on M by the formula

$$(3)\quad s_{(\alpha,u)}(\beta,v) = \begin{cases} (\alpha\cdot\beta, s_u^{[\alpha]}v) \text{ if } \alpha \cong \beta\ ;\ u,v\in M_{[\alpha]} \\ (\alpha\cdot\beta,\ v) \text{ if } \alpha \ncong \beta\ ;\ u\in M_{[\alpha]},\ v\in M_{[\beta]}. \end{cases}$$

Then $(M,\{s_x\})$ is a regular s-manifold composed of the components $(M_\alpha, \{s^\alpha_{x_\alpha}\})$ and with the index groupoid $(A,\cdot)$.

<u>Proof</u>. The formulas (3) are correct because $[\alpha\cdot\beta] = [\beta]$ for every $\alpha,\beta\in A$.

We have to prove

$$(\alpha,u)((\beta,v)\cdot(\gamma,w)) = ((\alpha,u)\cdot(\beta,v))\cdot((\alpha,u)\cdot(\gamma,w))$$

in the following 4 cases:

1) $\beta\cong\gamma\cong\alpha$ $\quad$ $u,v,w\in M_{[\alpha]}$
2) $\beta\cong\gamma,\ \alpha\ncong\gamma$ $\quad$ $u\in M_{[\alpha]};\ v,w\in M_{[\gamma]}$
3) $\beta\ncong\gamma,\ \alpha\cong\gamma$ $\quad$ $u,w\in M_{[\gamma]},\ v\in M_{[\beta]}$
4) $\beta\ncong\gamma,\ \alpha\ncong\gamma$ $\quad$ $u\in M_{[\alpha]},\ v\in M_{[\beta]},\ w\in M_{[\gamma]}$.

For the sake of brevity, we make the following denotations:

$$L = (\alpha,u)\cdot((\beta,v)\cdot(\gamma,w))$$
$$R = ((\alpha,u)\cdot(\beta,v))\cdot((\alpha,u)\cdot(\gamma,w))$$
$$(\alpha,u\cdot v) := (\alpha, s_u^{[\alpha]}v) \text{ if } \alpha\in A,\ u,v\in M_{[\alpha]}.$$

Ad 1) $\quad L = (\alpha,u)\cdot(\beta\cdot\gamma, v\cdot w) = (\alpha\cdot(\beta\cdot\gamma),\ u\cdot(v\cdot w))$

$\quad R = (\alpha\cdot\beta, u\cdot v)\cdot(\alpha\cdot\gamma, u\cdot w) = ((\alpha\cdot\beta)\cdot(\alpha\cdot\gamma), (u\cdot v)\cdot(u\cdot w))$

According to the regularity of $M_{[\alpha]}$ and A, we have $L = R$.

Ad 2) $\quad L = (\alpha,u)\cdot(\beta\cdot\gamma, v\cdot w) = (\alpha\cdot(\beta\cdot\gamma), v\cdot w)$

because $\alpha\ncong\beta\cdot\gamma$

$R = (\alpha\cdot\beta, v)\cdot(\alpha\cdot\gamma, w) = ((\alpha\cdot\beta)\cdot(\alpha\cdot\gamma),\ v\cdot w)$

because $\alpha\ncong\beta,\ \alpha\ncong\gamma,\ \alpha\cdot\beta\cong\alpha\cdot\gamma$.

Hence L = R.

Ad 3) $L = (\alpha,u)\cdot(\beta\cdot\gamma,w) = (\alpha\cdot(\beta\cdot\gamma),u\cdot w)$

because $\beta \not\cong \gamma$, $\alpha \cong \beta\cdot\gamma$

$R = (\alpha\cdot\beta,v)\cdot(\alpha\cdot\gamma,u\cdot w) = ((\alpha\cdot\beta)\cdot(\alpha\cdot\gamma),u\cdot w)$

because $\alpha \not\cong \beta$, $\alpha\cdot\beta \not\cong \alpha\cdot\gamma$.

Hence L = R.

Ad 4) $L = (\alpha,u)\cdot(\beta\cdot\gamma,w) = (\alpha\cdot(\beta\cdot\gamma),w)$

$R = (\alpha\cdot\beta,v)\cdot(\beta\cdot\gamma,w) = ((\alpha\cdot\beta)\cdot(\alpha\cdot\gamma),w)$

because $\alpha\cdot\beta \not\cong \beta\cdot\gamma$.

Hence L = R.

This completes the proof of the regularity.

Finally, $s_{(\alpha,u)}(\alpha,v) = (\alpha,u\cdot v)$ holds for each $\alpha \in A$, and hence the α-component of $(M, \{s_x\})$ is isomorphic to $(M_\alpha, \{\bar{s}^{\alpha}_{x_\alpha}\})$.

<u>Special case</u>. If the groupoid $(A, \cdot)$ is trivial in the sense that $\alpha\cdot\beta = \beta$ for each $\alpha, \beta \in A$, we get $\alpha \cong \beta$ if and only if $\alpha = \beta$ in A.

Hence

$$s_u v = \overset{\alpha}{s}_u v \quad \text{for} \quad u,v \in M_\alpha$$

$$s_u v = v \quad \text{for } u \in M_\alpha,\ v \in M_\beta \text{ and } \alpha \neq \beta$$

and this is the generalization of example 1.

§ 2. In the second part of this article we shall characterize the regular s-manifolds of 2 components and also generalise example 4 from [1]. (A classification of these s-manifolds remains an open problem.)

Let $(M, \{s_x\})$ be an arbitrary regular s-manifold. Let $G(M)$ denote the free group generated by all elements $x \in M$ (the

multiplication will be denoted by the symbol $\circ$). Let $H(M, \{s_x\})$ be the set of all elements of $G(M)$ of the form $x^{-1} \circ (s_x y)^{-1} \circ x \circ y$, and let $N(M, \{s_x\})$ be the subgroup of $G(M)$ generated by the set $\bigcup_{g \in G} g \circ H \circ g^{-1}$.
Clearly, $N(M, \{s_x\})$ is a normal subgroup of G.

(4) Let $p: G(M) \longrightarrow \mathrm{Aut}(M, \{s_x\})$

be the group homomorphism determined by the values $p(x) = s_x$, $x \in M$. Then $N(M, \{s_x\})$ belongs to the kernel of p, and p induces a homomorphism

(5) $\tau: G(M)/N(M, \{s_x\}) \longrightarrow \mathrm{Aut}(M, \{s_x\})$.

The image of the map p is a subgroup $G(M, \{s_x\}) \subset \mathrm{Aut}(M, \{s_x\})$ generated by all symmetries s_x, $x \in M$. Also, the restriction of p to $M \subset G(M)$ is a smooth map.

<u>Definition 3</u>. Let $(M, \{s_x\})$ be a regular s-manifold, and H an arbitrary Lie group. A homomorphism $\varphi: G(M) \longrightarrow H$ is said to be regular if the normal subgroup $N(M, \{s_x\}) \subset G(M)$ belongs to the kernel of φ, and the restriction $\varphi/_M$ is smooth.

Now we get the following

<u>Theorem</u>. Let $(M_1, \{s_x^1\})$, $(M_2, \{s_y^2\})$ be connected regular s-manifolds. All regular s-manifolds $(M_1 \vee M_2, \{s_z\})$ composed of $(M_1, \{s_x^1\})$ and $(M_2, \{s_y^2\})$ are in one-to-one correspondence with the pairs (φ, ψ) of a regular group homomorphism

$$
(6) \qquad \begin{aligned} \varphi &: G(M_1) \longrightarrow \mathrm{Aut}(M_2, \{s_y^2\}) \\ \psi &: G(M_2) \longrightarrow \mathrm{Aut}(M_1, \{s_x^1\}) \end{aligned}
$$

such that it holds

$$\text{(7)}\quad \begin{aligned} s^1_x \circ \psi(y) \circ (s^1_x)^{-1} &= \psi(\varphi(x)(y)) \\ s^2_y \circ \varphi(x) \circ (s^2_y)^{-1} &= \varphi(\psi(y)(x)) \end{aligned} \qquad x \in M_1,\ y \in M_2$$

<u>Proof</u>

A. Let $(M, \{s_z\})$ be a regular s-manifold which is composed of $(M_1, \{s^1_x\})$ and $(M_2, \{s^2_y\})$. Because $s_z \in \mathrm{Aut}(M, \{s_z\})$ for each $z \in M$, then $s_z|_{M_i} \in \mathrm{Aut}(M_i, \{s^i_{x_i}\})$ for $i = 1,2$ (and each $z \in M$). Hence we get group homomorphisms

$\pi_i : G(M, \{s_z\}) \longrightarrow \mathrm{Aut}(M_i, \{s^i_{x_i}\})$, $i = 1,2$

by the rule: $\pi_i(g) = g|_{M_i}$ for any $g \in G(M, \{s_z\})$. Further, we have canonical group injections

$e_i : G(M_i) \longrightarrow G(M)$ such that

$e_i\,[N(M_i, \{s^i_{x_i}\})] \subset N(M, \{s_z\})$ for $i = 1,2$.

Combining this with the regular group homomorphism $p: G(M) \longrightarrow G(M, \{s_z\})$, we obtain regular homomorphisms

$$\text{(8)}\quad h_{ij} = \pi_j \circ p \circ e_i : G(M_i) \longrightarrow \mathrm{Aut}(M_j, \{s^j_{x_j}\}),\ i,j = 1,2$$

Here h_{11}, h_{12} are the canonical homomorphisms p_1, p_2 of the form (4) and h_{12}, h_{21} are the wanted homomorphisms (6).

Finally, we obtain Formulas (7) from the relations

$$(s_x \circ s_y)|_{M_1} = (s_{s_x(y)} \circ s_x)|_{M_1}$$

$$(s_y \circ s_x)|_{M_2} = (s_{s_y(x)} \circ s_y)|_{M_2}$$

if we put $\varphi = h_{12}$, $\psi = h_{21}$.

B. Let be given connected regular s-manifolds $(M_1, \{s^1_x\})$, $(M_2, \{s^2_y\})$ and regular group homomorphisms φ, ψ of the form (6). Put $M = M_1 \vee M_2$ and define transformations s_z, $z \in M$, of M

as follows:

$$(9)\qquad \begin{aligned} &\text{For } x\in M_1 \text{ put } s_x|_{M_1} = s^1_x,\ s_x|_{M_2} = \varphi(x)\\ &\text{For } y\in M_2 \text{ put } s_y|_{M_1} = \psi(y),\ s_y|_{M_2} = s^2_y \end{aligned}$$

It is sufficient to prove the regularity of $\{s_z\}$.

a) $(s_x\circ s_{x'})|_{M_1} = s^1_x\circ s^1_x$, $(s_y\circ s_{y'})|_{M_2} = s^2_y\circ s^2_{y'}$;

and the regularity follows from the regularity of the components.

b) $$(s_x\circ s_{x'})|_{M_2} = \varphi(x)\circ\varphi(x') = \varphi(x\circ x') = \varphi(s_x(x')\circ x) = \varphi(s_x(x'))\circ\varphi(x) = s_{s_x(x')}\circ s_x|_{M_2} .$$

Similarly,

$(s_y\circ s_{y'})|_{M_1} = (s_{s_y(y')}\circ s_y)|_{M_1}$ follows from the regularity of ψ.

c) $(s_x\circ s_y)|_{M_1} = s^1_x\circ\psi(y) = \psi[\varphi(x)(y)]\circ s^1_x = s_{s_x(y)}\circ s_x|_{M_1}$

and $(s_y\circ s_x)|_{M_2} = (s_{s_y(x)}\circ s_y)|_{M_2}$ according to (7).

d) $(s_x\circ s_y)|_{M_2} = \varphi(x)\circ s^2_y = s^2_{\varphi(x)y}\circ\varphi(x) = s_{s_x(y)}\circ s_x|_{M_2}$

$(s_y\circ s_x)|_{M_1} = s_{s_y(x)}\circ s_y|_{M_1}$

because $\varphi(x)$, $\psi(y)$ are automorphisms of $(M_2, \{s^2_y\})$, $(M_1, \{s^1_x\})$ respectively.

Example. Let $(M, \{s_z\}) = (N, \{s^1_x\})\times(P, \{s^2_y\})$ be a direct product of s-manifolds, $\pi_1: M\to N$, $\pi_2: M\to P$ the projections. We define a regular s-manifold $(M\vee N, \{\bar{s}_u\})$ as follows:

$\varphi: G(N\times P) \longrightarrow Aut(N, \{s_x^1\})$ is defined by $\varphi(x,y) = s_x^1$

$\psi: G(N) \longrightarrow Aut(N\times P, \{s_x^1 \times s_y^2\})$ is defined by $\psi(x) = s_x^1 \times$ $\times\, id_P$, where $(s_x^1\times id_P)\ (x',y) = (s_x^1(x'),y)$ on $N\times P$.

We check the identities (7).

a) $L = s_{x'}^1 \circ \varphi(x,y) \circ (s_{x'}^1)^{-1} = s_{x'}^1 \circ s_x^1 \circ (s_{x'}^1)^{-1} = s^1_{s^1_{x'}(x)}$

$R = \varphi[\psi(x')(x,y)] = \varphi(s^1_{x'}\ x,y) = s^1_{s^1_{x'}(x)}$.

b) $L = s_{(x,y)} \circ \psi(x') \circ s^{-1}_{(x,y)} = s_x^1 \circ s_{x'}^1 \circ (s_x^1)^{-1} \times id_P$

$R = \psi[\varphi(x,y)(x')] = \psi(s_x^1(x')) = s^1_{s^1_x(x')} \times id_P$.

Let us write the explicit formula for the composed s-manifold $(M\vee N, \{\bar{s}_u\})$:

$$\bar{s}_{(x,y)}\big|_N = s_x^1 \quad \text{for } x\in N,\ y\in P$$

$$\bar{s}_x\big|_M = \bar{s}_x\big|_{N\times P} = s_x^1 \times id_P \text{ for } x\in N.$$

This generalizes example 4 from [1].

References

[1] KOWALSKI Oldřich: Generalized Symmetric Spaces, Lecture Notes in Mathematics, 805, Springer- Verlag Berlin-Heidelberg-New York, 1980.

[2] KOWALSKI Oldřich: Riemannian manifolds with general symmetries, Math. Zeitschrift 136(1974), 135-150.

[3] KOWALSKI Oldřich: Smooth and affine s-manifolds, Periodica Mathematica Hungarica 8(1977), 299-311.

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