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ON BIREGULAR AND REGULAR RINGS

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Abstract: A generalization of injectivity, noted Tp-injectivity, is introduced to study biregular rings and von Neumann regular rings.

Key words: Biregular, von Neumann regular, Tp-injective, p-injective, V-rings.

Classification: 16A15, 16A30, 16A32, 16A52

Throughout, A represents an associative ring with identity and A-modules are unitary. A left A-module M is called p-injective if, for any principal left ideal P of A and any left A-homomorphism $g:P \longrightarrow M$, there exists $y \in M$ such that $g(b) = by$ for all $b \in P$. In [10] through [14], left p-injective rings and p-injective modules are considered. Semi-group analogues of ring results on injectivity and p-injectivity are investigated in [6] and [7]. Since a few years, biregular rings, regular rings, V-rings and their generalizations are studied by various authors (cf. for example, the bibliography of [3],[4]). The purpose of this note is to study biregular and regular V-rings in terms of the following generalization of injectivity:

Definition. A left A-module M is called Tp-injective (two-sided ideal p-injective) if, for any ideal I of A, $a \in A$, any left A-homomorphism $g: Ia \longrightarrow M$, there exists $y \in M$ such that $g(ta) = tay$ for all $t \in I$. (An ideal of A will always mean a two-sided ideal.)

Obviously, Tp-injectivity implies p-injectivity. Note that if A is a simple ring, then a left A-module is Tp-injective iff it is p-injective. (Simple self-injective regular rings need not be Artinian (K.R. Goodearl).)

Write "A satisfies (∗)" if every proper ideal of A is a Tp-injective left A-module. Recall that A is biregular if, for any $a \in A$, the ideal AaA is generated by a central idempotent. As usual,

(1) A is called a left V-ring if every simple left A-module is injective;

(2) A is fully left idempotent if every left ideal is idempotent;

(3) A is reduced if it contains no non-zero nilpotent element.

(4) A is ELT(MELT) if every essential (maximal essential) left ideal is an ideal of A [12].

We first derive a few properties of rings satisfying (∗).

Proposition 1. _Let_ A _satisfy_ (∗). Then

(1) _For any factor ring_ B _of_ A, _every ideal of_ B _is generated by a central idempotent_. _In particular_, A _is a biregular fully right idempotent ring_;

(2) _Any prime factor ring of_ A _is simple_.

Proof. (1) For the first part, it is sufficient to prove that every ideal T of A is generated by a central idempotent. If $i: T \longrightarrow T$ is the identical map, there exists $u \in T$

such that $i(t) = tu$ for all $t \in T$. In particular, $u = i(u) = u^2$ and $T = Au$. Thus the left singular ideal and the Jacobson radical of A are both zero. Therefore A is semi-prime which implies that u is a central idempotent, whence A is biregular. Now for any $a \in A$, if $AaA = A$, then $a \in (aA)^2$. If $AaA \neq A$, $j: Aa \longrightarrow AaA$ the canonical injection, then there exists $b \in AaA$ such that $a = j(a) = ab \in (aA)^2$ again, which proves that A is fully right idempotent.

(2) follows from the fact that any non-zero ideal in a prime ring is left and right essential.

Corollary 1.1. If A satisfies (*), the centre of A is von Neumann regular.

Applying [1, Theorem 1] to Proposition 1, we get

Corollary 1.2. If A is a P.I. ring satisfying (*), then A is a regular left and right V-ring.

Corollary 1.3. If A is an indecomposable ring satisfying (*), then A is simple.

Corollary 1.4. Let A satisfy (*). Then (1) A is regular iff every primitive factor ring of A is regular; (2) If every primitive factor ring of A is MELT, then A is a unit-regular left and right V-ring whose prime factor rings are Artinian.

Proof. (1) Apply [4, Theorem 1.28] to Proposition 1(2). (2) Every prime factor ring of A is MELT simple and hence Artinian. Then A is regular by (1) which implies A unit-regular [4, Theorem 6.10]. A is a left and right V-ring by [3, Theorem 14].

Corollary 1.5. Let A be a directly finite left self-injective ring satisfying (∗). Then every prime factor ring of A is simple left self-injective.

(Apply [4, Theorem 9.32].)

Since a biregular ring is fully idempotent and any factor ring of a MELT ring is MELT, [4, Theorem 1.18 and Theorem 6.10] imply

Proposition 2. Let A be a MELT biregular ring. Then A is a unit-regular left and right V-ring whose prime factor rings are Artinian.

Corollary 2.1. If A is an ELT fully idempotent ring whose primitive factor rings are biregular, then A is a unit-regular left and right V-ring.

Proof. Any prime factor ring B of A is ELT fully idempotent which implies B primitive and hence Artinian by Proposition 2.

Rings whose left ideals are quasi-injective (called left q-rings) may be characterized as ELT left self-injective rings [5, Theorem 2.3]. For left self-injective rings in general, non-zero ideals need not contain non-zero central idempotents. However, we have

Remark 1. Let A be a left or right self-injective regular ring such that any prime factor ring is MELT. Then A is left and right self-injective biregular. Consequently, semiprime left q-rings are right self-injective biregular.

Remark 2. A left and right V-ring whose prime factor rings are MELT is a unit-regular ring such that the maximal left quotient ring coincides with the right one.

Let us now characterize rings whose left modules are Tp-injective. Following [8], a left A-module M is called semi-simple if the intersection of all maximal left submodules is zero. A is semi-simple Artinian iff every semi-simple left A-module is injective [8, Theorem 3.2].

Theorem 3. The following conditions are equivalent for a ring A:

(1) Every left A-module is Tp-injective;

(2) Every semi-simple left A-module is Tp-injective;

(3) Every essential left ideal of A is Tp-injective;

(4) A is a regular ring satisfying (*).

Proof. Obviously, (1) implies (2).

Assume (2). Then every semi-simple left A-module is p-injective which implies that A is von Neumann regular, whence every left ideal of A is semi-simple. Therefore (2) implies (3).

Assume (3). Since every essential left ideal is p-injective, then A is regular. If I is a proper ideal of A, there exists a complement left ideal C such that $I \oplus C$ is an essential left ideal. Since a direct summand of a Tp-injective left A-module is Tp-injective, then ${}_AI$ is Tp-injective and (3) implies (4).

Assume (4). Every ideal of A is a principal left ideal by Proposition 1. Then every left A-module, being p-injective, is Tp-injective which shows that (4) implies (1).

Corollary 3.1. If every ideal of A is generated by an element, then A is regular biregular iff every left A-module is Tp-injective.

Rings whose left ideals not isomorphic to $_AA$ are quasi-injective (resp. p-injective), noted wq (resp. WP) rings, are studied in [9] and [13]. Now call A a WTP ring (weak Tp-injective) if every left ideal not isomorphic to $_AA$ is Tp-injective. Simple regular rings and left principal ideal domains (written PID) are examples of WTP rings.

Since any ideal which is a Tp-injective left A-module is a direct summand of $_AA$, the next lemma then follows from [13, Lemma 1.1].

Lemma 4. If A is a WTP ring, then A is semi-prime with p-injective left socle such that any finitely generated left ideal or ideal of A is a principal projective left ideal.

Applying the proof of [13, Proposition 1.9], Proposition 1, Theorem 3 and Lemma 4, we get

Proposition 5. Let A be a WTP ring satisfying any one of the following conditions:

(1) A contains a central zero-divisor;

(2) There exists a proper ideal I such that A/I is a regular ring;

(3) A is a direct sum of two left ideals which are of infinite left Goldie dimension.

Then A is a regular ring whose left A-modules are Tp-injective.

Proposition 6. The following conditions are equivalent:

(1) A is either a left duo left PID or semi-simple Artinian;

(2) A is an ELT, WTP ring.

Proof. Obviously, (1) implies (2).

Assume (2). By Lemma 4, every essential left ideal is principal which implies that A is a principal left ideal ring. Then any left ideal not isomorphic to ${}_AA$ is injective. In particular, A is a wq-ring which implies that A is either a left PID or strongly regular left self-injective or has non-zero socle [9]. If A is a left PID, then any non-zero left ideal is essential which implies A left duo. If A is left self-injective, then every left ideal is injective which implies A semi-simple Artinian. Finally, if A has non-zero socle, then A is Artinian by [9, Lemma 1.5]. Thus (2) implies (1).

After considering regular rings satisfying (∗), we now look at WP-rings satisfying (∗).

Proposition 7. Let A be a WP-ring satisfying (∗). Then A is a WTP ring which is either simple or regular.

Proof. Apply [13, Lemma 1.3] to Proposition 1 and Corollary 1.3.

If A is fully right idempotent, then ${}_AA/T$ is flat for any ideal T of A. Lemma 4 then implies

Remark 3. If A is a WTP fully right idempotent ring, then every ideal of A is generated by a central idempotent. In particular, A is biregular.

We now characterize semi-simple Artinian rings in terms of WTP rings and rings satisfying (∗). ALD (almost left duo) rings are studied in [11] and [14].

Theorem 8. The following conditions are equivalent:

(1) A is semi-simple Artinian;

(2) Every essential left ideal of A is quasi-injective and

Tp-injective;

(3) A *is a left q-ring satisfying* (*);

(4) A *is a MELT ring satisfying* (*);

(5) A *is WTP ring with essential left socle*;

(6) A *is a MELT, WTP fully right idempotent ring*;

(7) A *is an ALD, WTP ring with non-zero socle*.

Proof. (1) implies (2) and (5) evidently.

Assume (2). Then any left ideal (being a direct summand of an essential left ideal) is quasi-injective and Tp-injective which shows that (2) implies (3).

(3) implies (4) by [5, Theorem 2.3].

Assume (4). If L is a proper essential left ideal, M a maximal left ideal containing L, then ${}_AM$ is Tp-injective which implies ${}_AM$ a direct summand of ${}_AA$. This contradiction proves that any left ideal is a direct summand of ${}_AA$ and (4) implies (1).

Assume (5). Let S be the left socle of A. If $S \neq A$, since S is an ideal, ${}_AS$ is a direct summand of ${}_AA$ which contradicts S essential. Thus S = A and (5) implies (6).

(6) implies (7) by Remark 3.

(7) implies (1) by [14, Lemma 1.1], Lemma 4 and Theorem 10 below.

Call A left Tp-injective if ${}_AA$ is Tp-injective.

Theorem 9. *The following conditions are equivalent:*

(1) A *is a left and right self-injective strongly regular ring*;

(2) A *is a left non-singular left* Tp-*injective ring such that every complement left ideal is an ideal*;

(3) A is a reduced left Tp-injective ring.

Proof. (1) implies (2) obviously.

(2) implies (3) by [10, Lemma 1].

Assume (3). Since A is reduced left p-injective, then A is strongly regular by [10, Theorem 1]. Therefore A is left self-injective and since A is strongly regular, then A is right self-injective. Thus (3) implies (1).

[14, Lemma 1.1] then implies

Corollary 9.1. The following conditions are equivalent:

(1) A is either semi-simple Artinian or left and right self-injective strongly regular;

(2) A is a semi-prime ALD left Tp-injective ring;

(3) A is a semi-prime ALD right Tp-injective ring.

Theorem 10. The following conditions are equivalent:

(1) A is a finite direct sum of division rings;

(2) Every ideal of A is a Tp-injective left A-module and every complement left ideal is an ideal;

(3) A is a reduced WTP ring with non-zero socle;

(4) A is a reduced WTP ring containing a non-zero p-injective left ideal.

Proof. (1) implies (2) evidently.

Assume (2). By Proposition 1 and Theorem 9, A is strongly regular. Then every left ideal is injective which implies A semi-simple Artinian. Since A is reduced, then (2) implies (3).

(3) implies (4) by [13, Proposition 1.4].

(4) implies (1) by [13, Corollary 1.6] and Remark 3.

We now consider Tp-injectivity in connection with continuous regular and Baer regular rings. Recall that (1) A is left continuous (in the sense of Y. Utumi) if (a) every left ideal isomorphic to a direct summand of ${}_AA$ is itself a direct summand of ${}_AA$ and (b) every complement left ideal is a direct summand of ${}_AA$; (2) A is a Baer ring if every left annihilator ideal is a direct summand of ${}_AA$; (3) A is quasi-Baer if the right annihilator of every ideal is a direct summand of ${}_AA$.

Proposition 11. (1) If A is a semi-prime ELT ring whose complement left ideals are Tp-injective, then A is left continuous regular;

(2) If A is an ELT ring whose left annihilator ideals are Tp-injective, then A is a Baer regular ring.

Proof. (1) If C is a complement left ideal of A, D a left ideal such that $L = C \oplus D$ is an essential left ideal, $h: : L \longrightarrow C$ the natural projection, then there exists $c \in C$ such that $h(u) = uc$ for all $u \in L$. In particular, $c = h(c) = c^2$ and $C = Ac$ is a direct summand of ${}_AA$. Since A is left p-injective, then any left ideal isomorphic to a direct summand of ${}_AA$ is principal p-injective and therefore a direct summand of ${}_AA$. This proves A left continuous. Now A semi-prime ELT implies A left non-singular whence A is left continuous regular.

(2) is similarly proved.

The proof of Proposition 11 yields

Remark 4. If A is a semi-prime ELT ring whose proper complement left ideals are Tp-injective, then A is either a left duo left Ore domain or a left continuous regular ring.

Looking back at Proposition 1, we see that a ring satisfying ($*$) is quasi-Baer. Also, if A satisfies ($*$) and A =

$= B \oplus C$, where B, C are ideals of A, then any ideal of B is generated by a central idempotent. [2, Theorem 3] and [10, Theorem 1] then yield

Proposition 12. If A is a left or right p-injective ring satisfying ($*$), then $A = B \oplus C$, where B is a finite direct sum of division rings and C is the minimal direct summand of ${}_AA$ containing the nilpotent elements of A.

Our last remark will follow from [9, Theorem 2.7] and Theorem 8.

Remark 5. A wq-ring satisfying ($*$) is either semi-simple Artinian or a simple left PID.

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