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PROVABILITY IN THE ALTERNATIVE SET THEORY

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Abstract: We demonstrate that every proof of finite length of a finite formula can be modified to a finite proof. We show which proofs can be replaced by finite proofs.

Key words: Alternative set theory, (finite) formula, (finite) proof.

Classification: Primary 03E70, 03H99
Secondary 03F99

In the alternative set theory (AST) we have two types of natural numbers: finite natural numbers which play the role of classical natural numbers and natural numbers (general, possibly infinite) which finiteness is only formal. In this sense we have two types of formulas (finite and general), two types of proofs, provabilities and consistences. Let us remind we use the word "finite" to express the notion constructed with respect to finite natural numbers (see [S 1] and [R 1]). Finite formulas have the main importance for us and we have been formulating main results on them; general formulas have an auxiliary value.

The aim of this article is to examine the connection between the provability and finite provability. Let φ be a finite formula (of some language) which is provable (provable means provable in the predicate calculus). Under what conditions is this formula finitely provable? Assume d the proof of φ . We may suppose the formulas in d do not contain other nonlogical symbols than these contained in φ . Thus all symbols in formulas in d are finite-ary. We have four possibilities:

1. The proof d has an infinite length and formulas in d are general (possibly infinite). In this case φ need not be finitely provable. Moreover the statement: "Every provable finite formula is finitely provable" is independent of the axioms of AST: when the axiom of elementary equivalence ($V \equiv FV$) is assumed, then the statement is true, and when we add the axiom $\neg \mathrm{Con}_{ZF_{fin}}$ (AST with this axiom is consistent, see [S 3]), then it is false.

2. d has an infinite length but contains only finite formulas. As there are only countably many finite formulas in the language of φ and d is a set, there is only a finite number of different formulas in d. Form d′ from d by omitting all formulas except their first occurence. Certainly d′ is finite proof of φ and φ is finitely provable.

3. d has a finite length but contains arbitrary formulas. We show as the main result of this article that φ is finitely provable in this case.

4. d has a finite length and contains only finite formulas. Then φ is finitely provable according to the definition.

Thus it remains to prove the following theorem:

<u>Theorem</u>. Let φ be a finite formula provable by a proof of a finite length. Then φ is finitely provable.

Proof. In the proof of φ there may be arbitrary variables, possibly x_α with infinite α. For the construction below we will understand these variables to be finite terms and consider that they do not break finiteness of a formula. At the end of our proof we show how to replace them by variables with finite indexes.

Let c be a fixed constant symbol in our language (if there is none, add one new constant symbol). Let τ be a term. Set

$$\tau^* = \begin{cases} \tau, & \text{if } \tau \text{ is finite} \\ c, & \text{otherwise.} \end{cases}$$

When φ is one of formulas $\neg\varphi_1$, $\varphi_1 \to \varphi_2$, $(\forall x)\varphi_1$, we say that φ_1, φ_2 resp. is an immediate subformula of φ. Let Γ be a finite set of formulas. In each formula φ in Γ, replace any term τ by τ^*. Let Γ_0 be a set of such obtained formulas from Γ. Proceed the construction by induction setting Γ_{n+1} the set of all formulas of Γ_n and all immediate subformulas of formulas of Γ_n. Take now $\Gamma^* = \Gamma_5 - \Gamma_4$, A the class of atomic formulas. For $\varphi \in \Gamma^* \cup (\Gamma_5 \cap A)$ define

$$\varphi^* \equiv \begin{cases} \varphi, & \text{if } \varphi \text{ is finite} \\ c = c, & \text{otherwise.} \end{cases}$$

Then define $(\neg\varphi)^* \equiv \neg\varphi^*$, $(\varphi_1 \to \varphi_2)^* = \varphi_1^* \to \varphi_2^*$, $((\forall x)\varphi)^* \equiv (\forall x)\varphi^*$. We prove by induction that for any $\varphi \in \Gamma_5$, the formula φ^* is defined. Let $\varphi \in \Gamma_5$. If φ is atomic then φ^* is defined. Let φ be not atomic and let for at least one immediate subformula φ' of φ have $\varphi' \notin \Gamma_5$. Then $\varphi \in \Gamma^*$ and φ^* is defined. If φ' is any immediate subformula of $\wp$ so that $\wp' \in \Gamma_5$, then by induction hypothesis $\wp'^*$ is defined; thus $\wp^*$ is defined in the remaining case.

Notice that any $\wp^*$ is finite and if $\wp$ is finite then $\wp^*$ and $\wp$ are equal.

Let $\varphi_1, \ldots, \varphi_n$ be the proof of φ, set $\Gamma = \{\varphi_1, \ldots \ldots, \varphi_n\}$. We show $\varphi_1^*, \ldots, \varphi_n^*$ is a proof of φ. If φ_i is one of axioms (cf. e.g. [V-H] § 2 ch. I)

$\psi_1 \longrightarrow (\psi_2 \longrightarrow \psi_1)$

$(\psi_1 \longrightarrow (\psi_2 \longrightarrow \psi_3)) \longrightarrow ((\psi_1 \longrightarrow \psi_2) \longrightarrow (\psi_1 \longrightarrow \psi_3))$

$(\neg \psi_1 \longrightarrow \neg \psi_2) \longrightarrow (\psi_2 \longrightarrow \psi_1)$

$(\forall x)(\psi_1 \longrightarrow \psi_2) \longrightarrow (\psi_1 \longrightarrow (\forall x) \psi_2)$

$(\forall x) \psi(x) \longrightarrow \psi(\tau)$

then φ_i^* is again an axiom of the same type. If φ_i is an identity or equality axiom, then it is a finite formula and φ_i^* is φ_i (φ_n is a finite formula, as above we assume all the φ_i's are in the language of φ_n). If φ_i is derived by the rule modus ponents from φ_j, $\varphi_j \longrightarrow \varphi_i$, then φ_i^* is derived from φ_j^*, $\varphi_j^* \longrightarrow \varphi_i^*$. If $\varphi_i \equiv (\forall x) \varphi_j$ is derived by the generalization rule from φ_j, then $\varphi_i^* \equiv (\forall x) \varphi_j^*$ is derived from φ_j^*. Every φ_i^* is a finite formula with the exception that it may contain variables with infinite indexes. The number of such variables in all φ_i^*'s is finite, thus we can replace them by finite variables (variables with finite indexes). This completes the proof.

In the theorem we considered the provability only in the predicate calculus. But it is easy to extend our result for provability in some concrete theories. The only requirement

we need is: if ψ is an axiom of the theory then ψ^* is also an axiom. It is obvious when ψ is a finite formula (it is the case of e.g. Gödel-Bernays set theory). But the extension of our result is also true for some theories having infinite axioms in its formalization, e.g. for Peano arithmetics, AST, Zermelo-Fraenkel set theory and others. These theories have axioms which are finite formulas and axiom-schemas, e.g. the schema $(\exists y)(\forall z)(z \in y \equiv \varphi(z,x_1,\dots,x_\alpha))$. If $\psi \in \Gamma_0$ denotes this formula then ψ^* is the formula
$(\exists y)(\forall z)(z \in y \equiv (\varphi(z,x_1,\dots,x_\alpha))^*)$
which is also an axiom of the same type. (For this reason it would be sometimes necessary to stop the construction of Γ_m's at some bigger n than n = 5 as we made above.)

Thus we see that for some common metamathematical theories T, if $\overline{T}$ and $\mathcal{T}$ is its formalization and finite formalization in AST respectively (see [S 1]), then if a finite formula φ is provable in $\overline{T}$ with a proof of finite length then $\mathcal{T} \vdash_F \varphi$.

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