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A SENTENCE THAT IS DIFFICULT TO INTERPRET
Vítězslav ŠVEJDAR

Abstract: A ZF-sentence \( \phi \) is found such that (ZF + \( \phi \)) is not interpretable in ZF, (GB + \( \phi \)) is not interpretable in GB, but (ZF + \( \phi \)) is interpretable in GB.

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Introduction. In 1972 Hájková and Hájek constructed an arithmetical sentence \( \phi \) such that (ZF + \( \phi \)) is relatively interpretable in ZF but (GB + \( \phi \)) is not relatively interpretable in GB ([2]). If we denote \( I_{ZF} \) and \( I_{GB} \) the sets of all sentences \( \phi \) such that (ZF + \( \phi \)) is relatively interpretable in ZF and (GB + \( \phi \)) is relatively interpretable in GB respectively, the result in [2] shows that \( I_{ZF} - I_{GB} \) is non-empty. In 1976 Solovay proved that also \( I_{GB} - I_{ZF} \) is non-empty ([4]). The relation between \( I_{ZF} \) and \( I_{GB} \) is further analysed in [1]. In the present paper we shall use the methods of [2] and [4] to obtain the following result.

Theorem. There exists a sentence \( \phi \) such that \( \phi \notin I_{ZF} \cup I_{GB} \) but (ZF + \( \phi \)) is relatively interpretable in GB.
Preliminaries and Solovay's provability predicates. We deal with metamathematics formalized within Peano arithmetic. Formulas and terms are identified with their Gödel numbers. 

\( \text{Con}(\varphi) \) is the usual consistency statement for a formula \( \varphi(x) \), \( \text{Intp}(z,x) \) expresses that \( z \) is a sentence and \( x \) is an interpretation of \( \text{GB} + z \) in \( \text{GB} \), where interpretation includes both translations of atomic formulas and proofs (in \( \text{GB} \)) of translated axioms (of \( \text{GB} + z \)), see [1] and [2]. \( \text{ZF} \upharpoonright n \) is the finite set of all axioms of \( \text{ZF} \) which are less than \( n \). In arithmetic, \( \text{zf} \) is the natural definition of all formal axioms of \( \text{ZF} \), in other words, \( \text{zf}(x) \) is the natural biniumeration of \( \text{ZF} \).

For a theory \( T \) in a language \( L \) let \( T_C \) be the conservative Henkin extension of \( T \) formulated in \( L_C \). Let \( \Delta(L) \) be the set of all closed instances (in \( L_C \)) of logical axioms, of axioms of identity and equality and of Henkin axioms ([3]). A sentence \( \varphi \) of \( L \) is provable in \( T \) if and only if it is a tautological consequence of \( \Delta(L) \cup T \) (see [3], p. 49). In the present paper \( L \) is the language of \( \text{ZF} \) while \( T \) is \( \text{ZF} \) or the predicate calculus for \( L \).

A function \( s \) associating 0 or 1 with every \( L_C \)-sentence less than \( n \) is a generalized satisfactory sequence on \( n \) if

1. \( s \) preserves logical connectives
2. \( s(\varphi) = 1 \) for every \( \varphi \in \Delta(L) \).

A function \( s \) is a satisfactory sequence on \( n \) if, in addition,

3. \( s(\varphi) = 1 \) for every \( \varphi \in \text{ZF} \).

The notion of satisfactory sequence is immediately formalized in arithmetic. Now let us define the formalized Solovay's provability predicates as follows:
Let \( \phi \) be a sentence in \( L^* \) then

(i) \( \text{Pr}(\phi) \iff \phi \) is provable in the predicate calculus,

(ii) \( \text{Pr}(\phi) \iff \phi \) is provable in \( ZF \).

Satisfaction relations. In \( GB + V = L \) we are able to define the partial satisfaction relations for formulas in \( L_\omega \). The axiom \( V = L \) is required for the definition of values of Henkin constants. For a more detailed treatment of satisfaction relations see [4] or [1].

A class \( Z \) is a satisfaction relation on \( j \) (in symbols \( \text{Tr}(Z,j) \)) if \( Z \) is a function defined on all pairs \( \langle a,u \rangle \) where \( u: \omega \to V \) is an evaluation of variables and \( a \) is a term or a formula in \( L_\omega \), \( a < j \). If \( a \) is a term, \( Z \) associates with it its "correct" value under \( u \), if \( a \) is a formula, \( Z \) associates with it its truth value 0 or 1. The inductive (Tarski's) conditions determine the values of \( Z \) uniquely. A number \( j \) is occupable (in symbols \( \text{Ocp}(j) \)) if there exists a satisfaction relation on \( j \). Satisfaction relations have the following properties:

Lemma (\( GB + V = L \)). (i) If \( \text{Ocp}(j) \), then the satisfaction relation on \( j \) is unique.
(ii) \{ j; Ocp(j) \} is a cut, i.e. it is closed under < and +1 but \{ j; Ocp(j) \} = \omega is unprovable.

(iii) If \varphi is a sentence of L then
\[ \vdash \text{Tr}(Z,j) & \& \varphi < j \rightarrow ( \varphi \equiv Z(\varphi,.) = 1) \].

(iv) If \text{Tr}(Z,j) then Z restricted to pairs \langle a,u \rangle where a is a sentence gives a satisfactory sequence on j.

The construction. We are now ready to define our sentence \varphi and prove its properties. \varphi is defined using the self-reference theorem as follows:
\[ \varphi \equiv \forall x,y \,(\text{Intp}(\varphi,x) & \text{Prf}(\varphi,y) & (&(zf \uparrow x) \rightarrow \neg \varphi ) \langle y \rightarrow \\
& \text{Prf}_o(&(zf \uparrow x) \rightarrow \neg \varphi, y)). \]

First, let us prove that (GB + \varphi) is not interpretable in GB. Assume the contrary. Then \text{Intp}(\varphi,x) has some standard witness \overline{m}. Let us denote \overline{d} = \&(zf \uparrow \overline{m}) \rightarrow \neg \varphi. Then
\[ (*) \vdash \varphi \rightarrow \forall y \,(\text{Prf}(\varphi,y) & \overline{d} < y \rightarrow \text{Prf}_o(&(zf \uparrow \overline{m}) \rightarrow \neg \varphi, y)). \]

By the essential reflexivity we have
\[ \vdash \varphi \rightarrow \text{Con}(zf \uparrow \overline{m} + \varphi). \]
That means, by (i) of our first lemma,
\[ (* *) \vdash \varphi \rightarrow \neg \text{Prf}_o(&(zf \uparrow \overline{m}) \rightarrow \neg \varphi). \]

By (\( \ast \ast \)) and (\( \ast \ast \ast \)) we have
\[ \vdash \varphi \rightarrow \forall y(\overline{d} < y \rightarrow \neg \text{Prf}(\varphi,y)). \]

But if \varphi is not provable on any level greater than \overline{d}, it is not provable at all. Hence by (ii) of the lemma
\[ \vdash \varphi \rightarrow \text{Con}(zf + \neg \varphi) \]
\[ \vdash \varphi \rightarrow \text{Con}(zf). \]
Hence \varphi implies Con(zf) which (being equivalent to Con(GB)) is not an element of \text{I}_{GB}. This is a contradiction with

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\( \varphi \in I_{\text{GB}} \)

For \( \varphi \notin I_{\text{ZF}} \) notice that the provability predicates are primitive recursive and \( \varphi \in \Pi_1 \). Since \( \varphi \) is unprovable, (ZF + \( \varphi \)) is not interpretable in ZF.

To interpret (ZF + \( \varphi \)) in GB it suffices to interpret (ZF + \( \varphi \)) in (GB + \( V = L + \neg \varphi \)). Let us proceed in the last theory. We have

\[ \exists x, y (\text{Intp}(\varphi, x) \land \text{Prf}(\varphi, y) \land (\&(zf \upharpoonright x) \rightarrow \overline{\neg \varphi}) < y \land \neg \text{Prf}_o (\&(zf \upharpoonright x) \rightarrow \overline{\neg \varphi}, y)) \]

As \( \neg \varphi \), by (iii) and (iv) of our second lemma, for every occupable \( j \) there exists a satisfactory sequence \( s \) on \( j \) such that \( s(\overline{\varphi}) = 0 \). Hence

\[ \forall j (0cp(j) \rightarrow \neg \text{Prf}(\varphi, j)) \]

and our \( y \) is nonoccupable. Also, since \( \text{Intp}(\overline{\varphi}, \cdot) \) has no standard witness, \( x \) is nonstandard.

Since \( \neg \text{Prf}_o (\&(zf \upharpoonright x) \rightarrow \overline{\neg \varphi}, y) \), by the definition of \( \text{Prf}_o \) there exists a generalized satisfactory sequence \( s \) on \( j \) such that \( s (\&(zf \upharpoonright x) \rightarrow \overline{\neg \varphi}) = 0 \). By the Solovay’s construction (see [4] or [1] for details) we can use \( s \) to construct an interpretation \( * \) of the language \( L \) such that for every sentence \( \psi \) in \( L \)

\[ \vdash \psi^* \equiv s(\overline{\varphi}) = 1 \]

But by the nonstandardness of \( x \) we have \( s(\overline{\varphi}) = 1 \) for every \( \psi \in \text{ZF} \) and also \( s(\overline{\varphi}) = 1 \) for our constructed \( \varphi \). This concludes our proof.

References


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