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A SENTENCE THAT IS DIFFICULT TO INTERPRET

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Abstract: A ZF-sentence φ is found such that $(ZF + \varphi)$ is not interpretable in ZF, $(GB + \varphi)$ is not interpretable in GB, but $(ZF + \varphi)$ is interpretable in GB.

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Introduction. In 1972 Hájková and Hájek constructed an arithmetical sentence φ such that $(ZF + \varphi)$ is relatively interpretable in ZF but $(GB + \varphi)$ is not relatively interpretable in GB ([2]). If we denote I_{ZF} and I_{GB} the sets of all sentences φ such that $(ZF + \varphi)$ is relatively interpretable in ZF and $(GB + \varphi)$ is relatively interpretable in GB respectively, the result in [2] shows that $I_{ZF} - I_{GB}$ is nonempty. In 1976 Solovay proved that also $I_{GB} - I_{ZF}$ is nonempty ([4]). The relation between I_{ZF} and I_{GB} is further analysed in [1]. In the present paper we shall use the methods of [2] and [4] to obtain the following result.

Theorem. There exists a sentence φ such that $\varphi \notin I_{ZF} \cup I_{GB}$ but $(ZF + \varphi)$ is relatively interpretable in GB.

Preliminaries and Solovay's provability predicates. We deal with metamathematics formalized within Peano arithmetic. Formulas and terms are identified with their Gödel numbers. $Con(\tau)$ is the usual consistency statement for a formula $\tau(x)$, $Intp(z,x)$ expresses that z is a sentence and x is an *interpretation* of (GB + z) in GB, where interpretation includes both translations of atomic formulas and proofs (in GB) of translated axioms (of (GB + z)), see [1] and [2]. $ZF \upharpoonright n$ is the finite set of all axioms of ZF which are less than n. In arithmetic, zf is the natural definition of all formal axioms of ZF, in other words, zf(x) is the natural binumeration of ZF.

For a theory T in a language L let T_c be the conservative *Henkin extension* of T formulated in L_c. Let $\Delta(L)$ be the set of all closed instances (in L_c) of logical axioms, of axioms of identity and equality and of Henkin axioms ([3]). A sentence φ of L is provable in T if and only if it is a tautological consequence of $\Delta(L) \cup T$ (see [3], p. 49). In the present paper L is the language of ZF while T is ZF or the predicate calculus for L.

A function s associating 0 or 1 with every L_c-sentence less than n is a *generalized satisfactory sequence* on n if

(1) s preserves logical connectives

(2) $s(\varphi) = 1$ for every $\varphi \in \Delta(L)$.

A function s is a *satisfactory sequence* on n if, in addition,

(3) $s(\varphi) = 1$ for every $\varphi \in ZF$.

The notion of satisfactory sequence is immediately formalized in arithmetic. Now let us *define* the formalized Solovay's *provability predicates* as follows:

$Prf_0(\varphi, x) \equiv \varphi < x$ and $s(\varphi) = 1$ for every generalized satisfactory sequence s on x

$Prf(\varphi, x) \equiv \varphi < x$ and $s(\varphi) = 1$ for every satisfactory sequence s on x

$Pr_0(\varphi) \equiv \exists x \, Prf_0(\varphi, x)$

$Pr(\varphi) \equiv \exists x \, Prf(\varphi, x)$.

We read $Prf(\varphi, x)$ as " φ is provable on level x". The provability predicates have the expected properties:

Lemma. Let φ be a sentence in L. Then

(i) $Pr_0(\varphi)$ iff φ is provable in the predicate calculus.

(ii) $Pr(\varphi)$ iff φ is provable in zf.

Satisfaction relations. In GB + V = L we are able to define the partial satisfaction relations for formulas in L_c. The axiom V = L is required for the definition of values of Henkin constants. For a more detailed treatment of satisfaction relations see [4] or [1].

A class Z is a satisfaction relation on j (in symbols Tr(Z,j)) if Z is a function defined on all pairs $\langle a,u \rangle$ where $u: \omega \longrightarrow V$ is an evaluation of variables and a is a term or a formula in L_c, $a < j$. If a is a term, Z associates with it its "correct" value under u, if a is a formula, Z associates with it its truth value 0 or 1. The inductive (Tarski's) conditions determine the values of Z uniquely. A number j is occupable (in symbols Ocp(j)) if there exists a satisfaction relation on j. Satisfaction relations have the following properties:

Lemma (GB + V = L). (i) If Ocp(j), then the satisfaction relation on j is unique.

(ii) $\{j;\ Ocp(j)\}$ is a cut, i.e. it is closed under $\prec$ and $+1$ but $\{j;\ Ocp(j)\} = \omega$ is unprovable.

(iii) If φ is a sentence of L then

$\vdash Tr(Z,j)\ \&\ \overline{\varphi} < j \rightarrow (\ \varphi \equiv Z(\overline{\varphi},\cdot) = 1)$.

(iv) If $Tr(Z,j)$ then Z restricted to pairs $\langle a,u\rangle$ where a is a sentence gives a satisfactory sequence on j.

<u>The construction</u>. We are now ready to define our sentence φ and prove its properties. φ is defined using the self-reference theorem as follows:

$$\vdash \varphi \equiv \forall x,y(Intp(\overline{\varphi},x)\ \&\ Prf(\overline{\varphi},y)\ \&\ (\&(zf \upharpoonright x) \rightarrow \overline{\neg\varphi}\ ) < y \rightarrow$$
$$\rightarrow Prf_0(\&(zf \upharpoonright x) \rightarrow \overline{\neg\varphi}\ ,y)).$$

First, let us prove that $(GB + \varphi\ )$ is <u>not interpretable</u> in GB. Assume the contrary. Then $Intp(\overline{\varphi},x)$ has some standard witness $\overline{m}$. Let us denote $\overline{d} = \&(zf \upharpoonright \overline{m}) \rightarrow \overline{\neg\varphi}$. Then

$(*)\quad \vdash \varphi \rightarrow \forall y(Prf(\overline{\varphi},y)\ \&\ \overline{d} < y \rightarrow Prf_0(\&(zf \upharpoonright \overline{m}) \rightarrow \overline{\neg\varphi}\ ,y))$.

By the essential reflexivity we have

$$\vdash \varphi \rightarrow Con(zf \upharpoonright \overline{m} + \overline{\varphi}).$$

That means, by (i) of our first lemma,

$(**)\quad \vdash \varphi \rightarrow \neg\ Pr_0(\&(zf \upharpoonright \overline{m}) \rightarrow \overline{\neg\varphi}\ )$.

By $(*)$ and $(**)$ we have

$$\vdash \varphi \rightarrow \forall y(\overline{d} < y \rightarrow \neg\ Prf(\overline{\varphi},y)).$$

But if $\overline{\varphi}$ is not provable on any level greater than $\overline{d}$, it is not provable at all. Hence by (ii) of the lemma

$$\vdash \varphi \rightarrow Con(zf + \overline{\neg\varphi})$$
$$\vdash \varphi \rightarrow Con(zf).$$

Hence φ implies Con(zf) which (being equivalent to Con(GB)) is not an element of I_{GB}. This is a <u>contradiction</u> with

$\varphi \in I_{GB}$.

For $\varphi \notin I_{ZF}$ notice that the provability predicates are primitive recursive and $\varphi \in \Pi_1$. Since φ is unprovable, $(ZF + \varphi)$ is <u>not interpretable</u> in ZF.

<u>To interpret</u> $(ZF + \varphi)$ in GB it suffices to interpret $(ZF + \varphi)$ in $(GB + V = L + \neg\varphi)$. Let us proceed in the last theory. We have

$$\exists x,y(\mathrm{Intp}(\overline{\varphi},x) \,\&\, \mathrm{Prf}(\overline{\varphi},y) \,\&\, (\&(zf \restriction x) \rightarrow \overline{\neg\varphi}) < y \,\& \\ \& \neg \mathrm{Prf}_0(\&(zf \restriction x) \rightarrow \overline{\neg\varphi},y)).$$

As $\neg\varphi$, by (iii) and (iv) of our second lemma, for every occupable j there exists a satisfactory sequence s on j such that $s(\overline{\varphi}) = 0$. Hence

$$\forall j\ (\mathrm{Ocp}(j) \rightarrow \neg \mathrm{Prf}(\overline{\varphi},j))$$

and our y <u>is nonoccupable</u>. Also, since $\mathrm{Intp}(\overline{\varphi},\cdot)$ has no standard witness, x <u>is nonstandard</u>.

Since $\neg \mathrm{Prf}_0(\&(zf \restriction x) \rightarrow \overline{\neg\varphi},y)$, by the definition of Prf_0 there exists a generalized satisfactory sequence s on j such that $s\ (\&(zf \restriction x) \rightarrow \overline{\neg\varphi}) = 0$. By the Solovay's construction (see [4] or [1] for details) we can use s to construct an interpretation $*$ of the language L such that for every sentence ψ in L

$$\vdash \psi^* \equiv s(\overline{\psi}) = 1.$$

But by the nonstandardness of x we have $s(\overline{\psi}) = 1$ for every $\psi \in ZF$ and also $s(\overline{\varphi}) = 1$ for our constructed φ . This concludes our proof.

R e f e r e n c e s

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