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ULTRAFILTERS OF SETS

A. SOCHOR, P. VOPĚNKA

Abstract: In the Alternative Set Theory we shall study ultrafilters of sets defining the following two characteristics of an ultrafilter $\mathcal{M}$: $\mu(\mathcal{M}) = \{\alpha;\ (\forall p)((p \hat{\trianglerighteq} \alpha \ \& \ \& \ \bigcup p_{\wedge} \in \mathcal{M}) \rightarrow p \cap \mathcal{M} \neq 0)\}$ and $\nu(\mathcal{M}) = \{\alpha;\ (\forall x \in \mathcal{M}) \neg \neg x \hat{\trianglerighteq} \alpha\}$. For any two cuts we discuss the existence of an ultrafilter such that its characteristics equal to given cuts.

Key words: Alternative Set Theory, ultrafilter, set-theoretically definable class, ω-complete ultrafilter.

Classification: 03E70, 03H99

In AST, we are going to deal with ultrafilters on the system of all set-theoretically definable classes (cf. ch. II [V]; this system is denoted by the symbol Sd_V). In the paper we restrict ourselves to ultrafilters containing a set. The theory of such ultrafilters essentially differs from the corresponding theory in ZF; in particular in AST we are able to construct on arbitrary infinite set ultrafilters which are "measures" (and moreover we can deal with additional properties which on one hand distinguish ultrafilters in AST and which on the other hand are equivalent in ZF).

For every set x there is exactly one natural number α such that there is a one-one mapping of x onto α which is a set (in symbols $x \mathrel{\hat{\approx}} \alpha$). We shall investigate the following two cuts which naturally characterize an ultrafilter $\mathcal{M}$:

$$\mu(\mathcal{M}) = \{\alpha;\ (\forall p)((p \mathrel{\hat{\lesssim}} \alpha \ \&\ \bigcup p \in \mathcal{M}) \rightarrow p \cap \mathcal{M} \neq 0)\}$$
$$\nu(\mathcal{M}) = \{\alpha;\ (\forall x \in \mathcal{M}) \neg\, x \mathrel{\hat{\lesssim}} \alpha\}$$

and we are going to show necessary and sufficient conditions for cuts R and S such that there is an ultrafilter $\mathcal{M}$ with $\mu(\mathcal{M}) = R \ \&\ \nu(\mathcal{M}) = S$. (Let us remind that in ZF if $\mathcal{M}$ is an ultrafilter which is a measure on a measurable cardinal $\varkappa$ then both correspoding characteristics equal to $\varkappa$.)

If $\mathcal{M}$ is a nontrivial ultrafilter on Sd_V containing a set then the routine calculation gives us the equalities $\mu(\mathcal{M}) = \{\alpha;\ (\forall p)((p \mathrel{\hat{\lesssim}} \alpha \ \&\ \bigcup p \in \mathcal{M}\ \&$ "p is pairwise disjoint"$) \rightarrow p \cap \mathcal{M} \neq 0)\} = \{\alpha;\ (\forall f)((\mathrm{dom}(f) \subseteq \alpha \ \&\ \mathrm{rng}(f) \subseteq \mathcal{M}) \rightarrow \bigcap \mathrm{rng}(f) \in \mathcal{M})\} = \{\alpha;\ (\forall f)((\mathrm{dom}(f) \subseteq \alpha \ \&\ \mathrm{rng}(f) \subseteq \mathcal{M}) \rightarrow \bigcap \mathrm{rng}(f) \neq 0)\}$.

In fact, defining for every p with $\bigcup p \in \mathcal{M}$ and every g which is a one-one mapping of p into α the function $f(\gamma) = \bigcup p - g^{-1}(\gamma)$ we get $\mathrm{dom}(f) \subseteq \alpha \ \&\ (p \cap \mathcal{M} = 0 \equiv \mathrm{rng}(f) \subseteq \mathcal{M}) \ \&\ \bigcap \mathrm{rng}(f) = 0$ and defining furthermore for every f with $\mathrm{dom}(f) \subseteq \alpha \ \&\ \mathrm{rng}(f) \subseteq \mathcal{M}$ the partition $p = \{\bigcap \mathrm{rng}(f)\} \cup \{\bigcap \{f(\beta);\ \beta \in \gamma\} - f(\gamma);\ 0 \in \gamma \in \alpha\}$ we get $\bigcup p = f(0) \in \mathcal{M} \ \&\ p \mathrel{\hat{\lesssim}} \alpha \ \&$ "p is pairwise disjoint" $\&\ (p \cap \mathcal{M} \neq 0 \equiv \bigcap \mathrm{rng}(f) \in \mathcal{M})$.

Let us recall that a nontrivial ultrafilter $\mathcal{M}$ containing a set is called ω-complete iff for every sequence $\{u_n;\ n \in FN\} \subseteq \mathcal{M}$ there is $u \in \mathcal{M}$ with $u \subseteq \bigcap \{u_n; n \in FN\}$. Furthermore X is a π-class (σ-class respectively) iff there is a sequence $\{X_n; n \in FN\} \subseteq Sd_V$ with $X = \bigcap\{X_n; n \in FN\}$ ($X = \bigcup\{X_n; n \in FN\}$ respectively). We are going to write $x \hat{\precsim} X$ iff there is a set-function of x into X and the symbol $x \hat{\not\precsim} X$ denotes $\neg x \hat{\precsim} X$. Evidently, if X is a cut then $x \hat{\precsim} X$ iff $(\exists \alpha \in X)\, x \hat{\approx} \alpha$.

<u>Theorem.</u> If $\mathcal{M}$ is a nontrivial ultrafilter on Sd_V containing a set then

(a) $\alpha, \beta \in \mu(\mathcal{M}) \rightarrow \alpha \cdot \beta \in \mu(\mathcal{M})$

(b) $(\alpha \in \mu(\mathcal{M})\ \&\ \beta \in \nu(\mathcal{M})) \rightarrow \alpha \cdot \beta \in \nu(\mathcal{M})$

(c) $FN \subseteq \mu(\mathcal{M}) \subseteq \nu(\mathcal{M}) \subset N$

(d) if $\nu(\mathcal{M})$ is a σ-class then $\mathcal{M}$ is not ω-complete

(e) if $\mathcal{M}$ is not ω-complete then $\mu(\mathcal{M}) = FN$

Proof. (a) If $\alpha, \beta \in \mu(\mathcal{M})$ and if $\bigcup\{u_{\gamma,\delta}; \gamma \in \alpha\ \&\ \&\ \delta \in \beta\} \in \mathcal{M}$ then there is $\delta_0 \in \beta$ such that $\bigcup\{u_{\gamma,\delta_0}; \gamma \in \in \alpha\} \in \mathcal{M}$ and thus there is $\gamma_0 \in \alpha$ with $u_{\gamma_0,\delta_0} \in \mathcal{M}$.

(b) If $u \hat{\precsim} \alpha \cdot \beta\ \&\ \alpha \in \mu(\mathcal{M})\ \&\ \beta \in \nu(\mathcal{M})$ then there is a sequence $\{u_\gamma; \gamma \in \alpha\}$ such that $u_\gamma \hat{\precsim} \beta$ for every $\gamma \in \alpha$ and such that $\bigcup\{u_\gamma; \gamma \in \alpha\} = u$. Supposing $u \in \mathcal{M}$ we can choose $\gamma_0 \in \alpha$ with $u_{\gamma_0} \in \mathcal{M}$ which contradicts $u_{\gamma_0} \hat{\precsim} \beta \in \in \nu(\mathcal{M})$.

(c) Since $\mathcal{M}$ is an ultrafilter, we have $2 \in \mu(\mathcal{M})$ and thence using (a) we get $FN \subseteq \mu(\mathcal{M})$. If $u \in \mathcal{M}\ \&\ u \hat{\approx} \alpha \in \in \mu(\mathcal{M})$ then $\{\{x\}; x \in u\} \hat{\precsim} \alpha\ \&\ \bigcup\{\{x\}; x \in u\} \in \mathcal{M}$ and therefore $\mathcal{M}$ would be trivial. Since $\mathcal{M}$ contains a set, we have $\nu(\mathcal{M}) \subset N$.

(d) Assuming that $\nu(\mathcal{M})$ is a σ-class we can construct a sequence $\{\alpha_n; n \in FN\}$ so that $\bigcap\{\alpha_n; n \in FN\} = \nu(\mathcal{M})$. Thus there is a sequence $\{u_n; n \in FN\} \subseteq \mathcal{M}$ such that $u_n \hat{\not\approx} \alpha_n$

for every $n \in FN$. Moreover if $u \subseteq \bigcap \{u_n; n \in FN\}$ then $u \hat{\geq} \nu(\mathfrak{M})$ and therefore $u \notin \mathfrak{M}$; we have proved that $\mathfrak{M}$ is not ω-complete.

(e) Let us assume that $\mu(\mathfrak{M}) \neq FN$ and that a sequence $\{u_n; n \in FN\} \subseteq \mathfrak{M}$ is given. Put $P = \{\bigcap \{u_m; m \leq n\} - u_{n+1}; n \in FN\}$. By the prolongation axiom there is a pairwise disjoint set p so that $P \subseteq p \;\&\; \bigcup p = u_0 \in \mathfrak{M} \;\&\; p \hat{\geq} \mu(\mathfrak{M})$. Thus there is $u \in p \cap \mathfrak{M}$ and evidently $u \subseteq \bigcap \{u_n; n \in FN\}$, thus $\mathfrak{M}$ is ω-complete.

Using the inequality $\alpha + \beta \leq 2 \max(\alpha, \beta)$ we get the following result.

Consequence. If $\mathfrak{M}$ is a nontrivial ultrafilter on Sd_V containing a set then the cuts $\mu(\mathfrak{M})$ and $\nu(\mathfrak{M})$ are closed w.r.t. +.

We are going to show that for every cuts $R \subseteq S \subset N$ there are ultrafilters $\mathfrak{M}, \mathfrak{M}'$ such that $\mu(\mathfrak{M}) = \mu(\mathfrak{M}') = R \;\&\; \nu(\mathfrak{M}) = \nu(\mathfrak{M}') = S$ and $\mathfrak{M}$ is ω-complete and $\mathfrak{M}'$ is not ω-complete under the assumption that the existence of such ultrafilters is not excluded by the first theorem of the article. In particular let us note that if S is a σ-class then for every ultrafilter on Sd_V containing a set we have $\mu(\mathfrak{M}) = FN$ and $\mathfrak{M}$ cannot be ω-complete and therefore the following theorem solves this case fully.

Theorem. If a proper semiset S is a cut which is closed w.r.t. + then there is a nontrivial ultrafilter $\mathfrak{M}$ on Sd_V which is not ω-complete and such that $\nu(\mathfrak{M}) = S$.

Proof. Put $\mathfrak{N} = \{\alpha - x; \alpha \notin S \;\&\; x \hat{\geq} S\}$. Supposing $\beta \geq \alpha \notin S \;\&\; x, y \hat{\geq} S$ we get $(\alpha - x) \cap (\beta - y) = \alpha - (x \cup y)$

and $x \cup y \stackrel{\wedge}{\supseteq} S$. Therefore we are able to choose an ultrafilter $\mathfrak{M}$ on Sd_V such that $\mathfrak{N} \subseteq \mathfrak{M}$. If $x \stackrel{\wedge}{\supseteq} S$ then for every $\alpha \notin S$ we have $\alpha - x \in \mathfrak{M}$ and thence $x \notin \mathfrak{M}$, from which the inclusion $S \subseteq \nu(\mathfrak{M})$ follows. The converse inclusion is a trivial consequence of the formula $\alpha \notin S \rightarrow \alpha \in \mathfrak{M}$.

If S is a π-class then there is no ω-complete ultrafilter $\mathfrak{M}$ on Sd_V with $\nu(\mathfrak{M}) = S$ and hence we can assume up to the end of the proof that S is no π-class. Let $\mathfrak{M}_0$ be a nontrivial ultrafilter on Sd_V with $\nu(\mathfrak{M}_0) = S$ and let $\mathfrak{M}_1$ be an ultrafilter on Sd_V such that $\mathfrak{M}_1 \supseteq \{\alpha - x;\ \alpha \notin FN \ \&\ x \stackrel{\wedge}{\supseteq} FN\}$, moreover, we can assume $\vartheta \notin S \rightarrow \vartheta \in \mathfrak{M}_0$. We put

$$\mathfrak{M} = \{U \in Sd_V;\ (\exists u \in \mathfrak{M}_1)(\forall n \in (FN \cap u))(U''\{n\} \in \mathfrak{M}_0)\}$$

(cf. the definition of product of ultrafilters in ZF). For every $U, W \in Sd_V$ we have $\{n; (U \cap W)''\{n\} \in \mathfrak{M}_0\} = \{n; U''\{n\} \in \mathfrak{M}_0\} \cap \{n;\ W''\{n\} \in \mathfrak{M}_0\}$ and hence $U, W \in \mathfrak{M} \equiv (U \cap W) \in \mathfrak{M}$. According to the prolongation axiom for every $U \in Sd_V$ there are u_1, u_2 such that $(u_1 \cup u_2) \in \mathfrak{M}_1 \ \&\ u_1 \cap u_2 = 0 \ \&\ \{n; U''\{n\} \in \mathfrak{M}_0\} \subseteq u_1 \ \&\ \{n; (-U)''\{n\} \in \mathfrak{M}_0\} \subseteq u_2$. Since $U \in \mathfrak{M} \equiv u_1 \in \mathfrak{M}_1$ we get $U \in \mathfrak{M} \equiv (-U) \notin \mathfrak{M}$. Thus we have proved that $\mathfrak{M}$ is an ultrafilter on Sd_V.

Let $\alpha \notin FN \ \&\ \vartheta \notin S$. For every $n \in FN$ we put $u_n = \vartheta \times (\alpha - n)$, thus u_n is an element of $\mathfrak{M}$ for every $n \in FN$, of course, and if $u \subseteq \bigcap\{u_n; n \in FN\}$ then there is $\beta \notin FN$ so that $u \cap (\vartheta \times \beta)) = 0$, from which $u \notin \mathfrak{M}$ follows. We have shown that $\mathfrak{M}$ is not ω-complete.

If $u \stackrel{\wedge}{\supseteq} S$ then for every $n \in FN$ we have $u''\{n\} \stackrel{\wedge}{\supseteq} S$ and hence $u''\{n\} \notin \mathfrak{M}_0$ and as a consequence we obtain $u \notin \mathfrak{M}$. Thus we have got the inclusion $S \subseteq \nu(\mathfrak{M})$. If $\sigma \notin S$ then for every $n \in FN$ there is $\gamma_n \notin S$ so that $\gamma_n \cdot n < \delta$ (because S is closed w.r.t. +) and moreover since we assume that S is no π-class we are able to choose $\gamma \notin S$ and $\beta \notin FN$ so that $\gamma \cdot \beta < \delta$. Evidently $\gamma \times \beta \in \mathfrak{M}$ and $\gamma \times \beta \hat{\preceq} \delta$ from which $\delta \notin \nu(\mathfrak{M})$ follows. Thus we have shown $\nu(\mathfrak{M}) \subseteq$ $\subseteq S$ which completes the proof.

The class Ω of ordinal numbers was defined in § 3 ch. II [V]. An ordinal number $\alpha \in \Omega$ is called a limit ordinal number iff there is no $\beta \in \Omega$ with $\alpha = \beta + 1$. To prove the main theorem of the paper we need the following two lemmas.

Lemma. Let $\{u_\alpha ; \alpha \in \Omega\} \subseteq P(w) - \{0\}$ be a descending sequence such that the formula $(\forall q)((\bigcup q = w \,\&\, q \hat{\preceq} FN) \rightarrow$ $\rightarrow (\exists \alpha \in \Omega)(\exists v \in q)(u_\alpha \subseteq v))$ holds. Then $\mathfrak{M} = \{U \in Sd_V;$ $(\exists \alpha \in \Omega)(u_\alpha \subseteq U)\}$ is an ω-complete ultrafilter on Sd_V (such that $w \in \mathfrak{M}$).

Proof. If $\{U_n; n \in FN\} \subseteq \mathfrak{M}$ then for every $n \in FN$ we can choose $\alpha_n \in \Omega$ so that $u_{\alpha_n} \subseteq U_n$. According to properties of Ω there is $\alpha \in \Omega$ with $(\forall n \in FN)(\alpha_n < \alpha)$ and for this α it is $u_\alpha \subseteq \bigcap \{U_n; n \in FN\} \,\&\, u_\alpha \in \mathfrak{M}$.

If $U \in Sd_V$ then $\{U \cap w, w - U\}$ is a partition of w and thence there is $\alpha \in \Omega$ such that either $u_\alpha \subseteq U \cap w \subseteq U$ or $u_\alpha \subseteq w - U \subseteq V - U$ holds. Thus we have proved that $U \in \mathfrak{M} \equiv$ $\equiv (V - U) \notin \mathfrak{M}$.

Lemma. Let $p \hat{\succeq} \alpha \cdot \beta \,\&\, q \hat{\preceq} \alpha \,\&\, \bigcup p \subseteq \bigcup q$ and let the formula $(\forall v \in p)(v \hat{\succeq} \gamma \cdot \alpha)$ hold. Then there is $u \in q$ with $\{v \in p; v \cap u \hat{\succeq} \gamma\} \hat{\succeq} \beta$.

Proof. If $(\forall u \in q)(\{v \in p; v \cap u \hat{\succeq} \gamma\} \hat{\prec} \beta)$ then $a =$ $= \{v \in p; (\exists u \in q)(v \cap u \hat{\succeq} \gamma)\} \hat{\prec} \beta \cdot \alpha$ and therefore we can choose $v \in p - a$. For this v we have $(\forall u \in q)(v \cap u \hat{\prec} \gamma)$ and this contradicts the assumption $\bigcup q \supseteq v$ since we would have $v \hat{\prec} \gamma \cdot \alpha$.

Theorem. If $FN \subseteq R \subseteq S \subset N$ are two cuts such that S is no π-class and such that the formula $(\forall \alpha \in R)(\forall \beta \in S)(\alpha \cdot \beta \in S) \& (\forall \alpha, \beta \in R)(\alpha \cdot \beta \in R)$ holds then there is an ω-complete ultrafilter $\mathcal{M}$ on Sd_V with $\mu(\mathcal{M}) = R \;\&\; \nu(\mathcal{M}) = S$.

Proof. Since S is no π-class, there is a descending sequence $\{\sigma_\alpha ; \alpha \in \Omega\}$ so that $\bigcap \{\sigma_\alpha ; \alpha \in \Omega\} = S$. Let w be a set such that there is $\vartheta \notin S$ with $w \hat{\approx} \vartheta$, let $\{q_\alpha ; \alpha \in \Omega\}$ be the class of all partitions q of w with $q \hat{\precsim} R$ and at the end let $\gamma_\alpha \hat{\approx} q_\alpha$. We are going to construct by induction a descending sequence $\{u_\alpha ; \alpha \in \Omega\}$ such that for every $\alpha \in \Omega$ the formula $u_\alpha \hat{\succ} S \,\&\, (\exists v \in q_\alpha)(u_{\alpha+1} \subseteq v)$ holds. We put $u_0 = w$. If we shall have such a sequence we shall put $\mathcal{M} = \{U \in Sd_V ; (\exists \alpha \in \Omega)(u_\alpha \subseteq U)\}$ and such a class will be an ω-complete ultrafilter according to a previous lemma.

Case 1. Let us suppose that $R = \{\alpha ; (\forall \beta \in S)(\alpha \cdot \beta \in S)\}$.

(a) Let for $\alpha \in \Omega$ the set u_α have been constructed, we want to construct the set $u_{\alpha+1}$. Evidently $u_\alpha \subseteq \bigcup q_\alpha$ and thus it is sufficient to choose $u_{\alpha+1}$ so that $(\exists v \in q_\alpha)(u_{\alpha+1} = u_\alpha \cap v \,\&\, (\forall v' \in q_\alpha)(u_{\alpha+1} \hat{\succsim} u_\alpha \cap v'))$. In fact, we have trivially $u_{\alpha+1} \subseteq u_\alpha$ and moreover, assuming that $u_{\alpha+1} \hat{\approx} \beta \in S$ we get $\beta \cdot \gamma_\alpha \in S$ (because $\gamma_\alpha \in R$) and hence we would obtain $u_\alpha \hat{\precsim} \beta \cdot \gamma_\alpha \in S$ which contradicts the induction hypothesis $u_\alpha \hat{\succ} S$; therefore we have proved the statement $u_{\alpha+1} \hat{\succ} S$.

(b) Let $0 \neq \alpha \in \Omega$ be a limit ordinal and let us assume that the sequence $\{u_\beta ; \beta \in (\alpha \cap \Omega)\}$ satisfying the properties in question has been constructed. Since S is no π-class, there is $\delta \notin S$ such that $(\forall \beta \in (\alpha \cap \Omega))(u_\beta \mathrel{\hat{\succeq}} \delta \,\&$ $\& \,\delta < \delta_\alpha)$ and according to the prolongation axiom we are able to choose u_α so that $u_\alpha \mathrel{\hat{\approx}} \delta$ and $u_\alpha \subseteq \bigcap \{u_\beta ; \beta \in$ $\in (\alpha \cap \Omega)\}$. (In detail: $\alpha \cap \Omega$ is countable and thence there is an increasing sequence $\{\alpha_n ; n \in FN\}$ such that $\bigcup \{\alpha_n ;$ $n \in FN\} = \bigcup (\alpha \cap \Omega)$ and we can choose f with $(\forall n \in FN)(f(n) =$ $= u_{\alpha_n}) \,\&\, (\forall \beta, \beta' \in \mathrm{dom}(f))(\beta < \beta' \rightarrow (f(\beta') \subseteq f(\beta) \,\&\, f(\beta) \mathrel{\hat{\succeq}}$ $\mathrel{\hat{\succeq}} \delta)$.) Moreover, choosing $\beta \in \mathrm{dom}(f) - FN$ we have $f(\beta) \subseteq$ $\subseteq \bigcap \{u_\gamma ; \gamma \in (\alpha \cap \Omega)\} \,\&\, f(\beta) \mathrel{\hat{\succeq}} \delta$ and therefore there is a one-one mapping of δ into $f(\beta)$; the range of such a mapping can serve as a set we look for).

We have $\nu(\mathfrak{M}) \subseteq S$ since for every limit ordinal $\alpha \in \Omega$ we have constructed u_α in such a way that $u_\alpha \mathrel{\hat{\preceq}} \delta_\alpha \,\&\, u \in$ $\in \mathfrak{M}$ and, on the other hand, the inclusion $S \subseteq \nu(\mathfrak{M})$ follows from the fact that $(\forall \alpha \in \Omega)(u_\alpha \mathrel{\hat{\succ}} S)$. Further from the construction we get $R \subseteq \mu(\mathfrak{M})$ and the converse inclusion is a consequence of the first theorem.

In the following two cases let $\varphi(p, u, \xi, \varsigma)$ denote the formula $\bigcup p \supseteq u \,\&\, \{v \in p ; u \cap v \mathrel{\hat{\succeq}} \xi\} \mathrel{\hat{\succeq}} \varsigma$. We can suppose that $\{\alpha ; (\forall \beta \in S)(\alpha \cdot \beta \in S)\} - R \neq 0$ and therefore we are able to fix ς in this class. According to $(\forall \xi' \in S)(\varsigma \cdot \xi' < \vartheta)$, we can choose moreover $\xi \notin S$ with $\varsigma \cdot \xi < \vartheta$ and furthermore using the last property, we can fix pairwise disjoint p such that $\bigcup p = w \,\&\, \varphi(p, w, \xi, \varsigma)$.

Case 2. We shall assume that R is no π-class. In this case there is a descending sequence $\{\varepsilon_\alpha ; \alpha \in \Omega\}$ so that $R =$ $= \bigcap \{\varepsilon_\alpha ; \alpha \in \Omega\}$.

Case 3. We shall suppose that R is no σ-class.

These two cases exhaust all possibilities since every semiset which is simultaneously π and σ is a set (cf. the last theorem of § 5 ch. II [V]), and R cannot be a set since it is closed w.r.t. +. We shall treat these two cases simultaneously and we are going to construct except a descending sequence of sets $\{u_\alpha ; \alpha \in \Omega\}$ even two descending sequences $\{\xi_\alpha ; \alpha \in \Omega\}$ and $\{\varrho_\alpha ; \alpha \in \Omega\}$ of natural numbers so that $(\forall \alpha \in \Omega)(\varphi(p, u_\alpha, \xi_\alpha, \varrho_\alpha) \& \xi_\alpha \notin S \& \varrho_\alpha \notin R)$ and if R is no σ-class we shall require moreover that for every $\alpha \in \Omega$ there is $\sigma \in R$ with $\varrho_\alpha \cdot \sigma > \varrho$. We put $u_0 = w$, $\xi_0 = \xi$ and $\varrho_0 = \varrho$.

(a) Let for $\alpha \in \Omega$ the sets u_α, ξ_α and ϱ_α with the required properties have been constructed. Let us define $\xi_{\alpha+1}$ and $\varrho_{\alpha+1}$ in such a way that the formula $\gamma_\alpha \cdot \xi_{\alpha+1} \leq \xi_\alpha < \gamma_\alpha \cdot (\xi_{\alpha+1} + 1) \& \gamma_\alpha \cdot \varrho_{\alpha+1} \leq \varrho_\alpha < \gamma_\alpha \cdot (\varrho_{\alpha+1} + 1)$ holds. We have $\gamma_\alpha \in R$ and hence according to the induction hypothesis $\xi_\alpha \notin S \& \varrho_\alpha \notin R$ we get $\xi_{\alpha+1} \notin S \& \varrho_{\alpha+1} \notin R$ and moreover if there is $\sigma \in R$ such that $\sigma \cdot \varrho_\alpha > \varrho$ then $\sigma \cdot (\gamma_\alpha + 1) \in R$ and $\sigma \cdot (\gamma_\alpha + 1) \cdot \varrho_{\alpha+1} > \sigma \cdot \varrho_\alpha > \varrho$.

Putting $\bar{p} = \{u_\alpha \cap v; v \in p \& u_\alpha \cap v \hat{\geq} \xi_{\alpha+1} \cdot \gamma_\alpha\}$ we obtain $\bar{p} \supseteq \{u_\alpha \cap v; v \in p \& u_\alpha \cap v \hat{\geq} \xi_\alpha\}$ from which $\bar{p} \hat{\geq} \varrho_\alpha \hat{\geq} \gamma_\alpha \cdot \varrho_{\alpha+1}$ follows. Furthermore according to $q_\alpha \hat{\approx} \gamma_\alpha$ and to the last lemma there is $u \in q_\alpha$ so that $\{v \in \bar{p}; v \cap u \hat{\geq} \xi_{\alpha+1}\} \hat{\geq} \varrho_{\alpha+1}$ and thence defining $u_{\alpha+1} = u \cap \cap u_\alpha$ we get $\varphi(p, u_{\alpha+1}, \xi_{\alpha+1}, \varrho_{\alpha+1})$.

(b) Let $0 \neq \alpha \in \Omega$ be a limit ordinal and let us suppose that we have constructed the sequences $\{u_\beta ; \beta \in (\alpha \cap \cap \Omega)\}$, $\{\xi_\beta ; \beta \in (\alpha \cap \Omega)\}$ and $\{\varrho_\beta ; \beta \in (\alpha \cap \Omega)$ satisfying the conditions in question. If R is no π-class then we can fix $\varrho_\alpha \in ((\bigcap\{\xi_\beta ; \beta \in (\alpha\cap\Omega)\} \cap \varepsilon_\alpha) - R$. If R is no σ-class then according to the induction hypothesis for every $\beta \in (\alpha \cap \Omega)$ we can choose $\sigma_\beta \in R$ with $\varrho_\beta \cdot \sigma_\beta > \varrho$. The class $\alpha \cap \Omega$ is countable and therefore there is $\sigma \in R$ such that $(\forall \beta \in (\alpha\cap\Omega))(\sigma_\beta < \sigma)$ and hence $(\forall \beta \in (\alpha \cap \Omega))(\varrho_\beta \cdot \sigma > \varrho)$. Let us fix ϱ_α so that $\varrho_\alpha \cdot \sigma > \varrho \geq (\varrho_\alpha - 1)\cdot \sigma$, thus even in this case ϱ_α is no element of R. The definition of ξ_α is in both cases the same as follows. Since $\varrho_\alpha < \varrho$ and since $\delta_\alpha \notin S$ there is $\xi' \notin S$ with $\varrho_\alpha \cdot \xi' < \delta_\alpha$ and thus we can fix ξ_α as an element of $(\bigcap\{\xi_\beta ; \beta \in (\alpha\cap\Omega)\} \cap \xi') - S$ because S is no π-class and because $\alpha \cap \Omega$ is countable.

By the prolongation axiom there is $u \subseteq \bigcap\{u_\beta ; \beta \in (\alpha \cap \Omega)\}$ so that $\varphi(p, u, \xi_\alpha, \varrho_\alpha)$ and according to the definition of φ, we are able to choose $u_\alpha \subseteq u$ in such a way that the formula $\varphi(p, u_\alpha, \xi_\alpha, \varrho_\alpha) \,\&\, u_\alpha \stackrel{\wedge}{\approx} \xi_\alpha \cdot \varrho_\alpha < \delta_\alpha \,\&\, \{v \in p; v \cap u_\alpha \neq 0\} \stackrel{\wedge}{\approx} \varrho_\alpha$ holds.

The statements $\nu(\mathfrak{M}) = S$ and $R \subseteq \mu(\mathfrak{M})$ can be proved exactly in the same way as in the first case.

The proof of the formula $\mu(\mathfrak{M}) \subseteq R$ is different in the cases we deal with; according to the definition of $\mu(\mathfrak{M})$, for every $\varepsilon \notin R$ we have to construct a partition $q \stackrel{\wedge}{\approx} \varepsilon$ with $q \cap \mathfrak{M} = 0$.

Case 2. If we put $p_\alpha = \{v \in p; u_\alpha \cap v \neq 0\}$ then $u_\alpha \subset \bigcup p_\alpha \subset \mathfrak{M}$ (α being an element of Ω). Let us suppose that there are $v \in p$ and $\alpha \in \Omega$ such that $u_\alpha \cap v$ is an element of $\mathfrak{M}$; then there would be $\beta \in \Omega$ with $u_\beta \subseteq u_\alpha \cap v$ but according to the construction we have $2 \stackrel{\wedge}{\approx} \{u \in p; u \cap u_\beta \neq 0\}$, which is a contradiction. Thus we have proved that for every $\alpha \in \Omega$ the classes p_α and $\mathfrak{M}$ are disjoint. Moreover, for every limit ordinal α we have $p_\alpha \mathrel{\hat{\approx}} \xi_\alpha < \varepsilon_\alpha$ and hence $\varepsilon_\alpha \notin \mu(\mathfrak{M})$ from which $R = \mu(\mathfrak{M})$ follows.

Case 3. To prove the inclusion $\mu(\mathfrak{M}) \subseteq R$ let $\varepsilon \notin R$ be given; without loss of generality we can suppose that $2\varepsilon < \xi$. Let γ be the minimal β so that $\beta \cdot \varepsilon \geq \xi$. At first we shall prove that for every $\alpha \in \Omega$ the formula $\xi_\alpha > \gamma$ holds. If it would not be true then there would be $\sigma \in R$ with $\sigma \cdot \gamma \geq \sigma \cdot \xi_\alpha \geq \xi$. Furthermore, $2\sigma \in R$ and hence $2\sigma < \varepsilon$ from which $\xi \leq \sigma \cdot \gamma = 2\sigma \cdot \frac{\gamma}{2} \leq \varepsilon \cdot \frac{\gamma}{2}$ would follow, but this contradicts the choice of γ since either γ or $\gamma + 1$ is even and both $\frac{\gamma}{2}$ and $\frac{\gamma+1}{2}$ are smaller than γ (because $\gamma \geq 2$).

Since $\gamma \cdot \varepsilon \geq \xi$, there is a partition $\overline{p}$ of w shich is coarser than p and such that $\overline{p} \mathrel{\hat{\approx}} \varepsilon$ & $(\forall u \in \overline{p})(\{v \in p; v \cap u \neq 0\} \mathrel{\hat{\preceq}} \gamma)$. To prove $\varepsilon \notin \mu(\mathfrak{M})$ it is sufficient to show that $\overline{p} \cap \mathfrak{M} = 0$. If the inclusion $u_\beta \subseteq u$ would hold for some limit $\beta \in \Omega$ and $u \in \overline{p}$ then $\gamma < \xi_\beta$ would contradict the formula $\{v \in p; v \cap u \neq 0\} \mathrel{\hat{\preceq}} \gamma$ & $\{v \in p; u_\beta \cap v \neq 0\} \mathrel{\hat{\approx}} \xi_\beta$ and this finishes the proof.

Reference

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