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Abstract: A continuous linear operator T which maps a Banach space X into a Banach space Y is said to be unconditionally converging (uc) if T maps weakly unconditionally converging (wuc) series into undconditionally converging (uc) series. X is said to have property V if for every Banach space Y, every uc operator $T:X \longrightarrow Y$ is weakly compact. We show that the space $C_o(S)$ and A(K) (with restricted conditions on K) have property V. (A(K) is the partially ordered Banach space of all continuous real-valued affine functions on K, a compact Choquet simplex.)

Key words: Banach space, unconditionally converging operator, weakly compact operator.

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N(X) is to denote JX (J is the natural map) plus all $\sigma(X'',X')$ limits of wuc series in X. N(X) is a subset of X" and JX = N(X) if and only if every wuc series in X is uc. The $\sigma(X',N(X))$ topology on X' is generated by polars of finite sets of N(X). Let S be separated locally compact space. $C_o(S)$ is the space of continuous functions x on S such that given $\varepsilon > 0$, the set $\{s\in S:|x(s)| \geq \varepsilon\}$ is conditionally compact in S. $C_o(S)$ is a Banach space with norm $\|x\|$ = $= \sup\{|x(s)|:s\in S\}$. M(S) is to denote the Banach space of bounded Radon measures on S, the norm being $\|\mu\| = \int d|\mu|$.

Recall that the dual of $C_0(S)$ may be identified with $M(S)$ by associating with each $\mu \in M(S)$ the linear form $x \longrightarrow \longrightarrow \int_S x \, d\mu$ on $C_0(S)$. If S is compact then $C_0(S)$ is the space $C(S)$. A proof that $C(S)$ has property V is given in [4].

<u>Theorem 1</u>. For any separated locally compact space S, $C_0(S)$ has property V.

<u>Proof</u>. Let $T: C_0(S) \longrightarrow Y$ be a uc operator for an arbitrary Banach space Y. Grothendieck (Theorem 6 of [1]) proved that T is weakly compact if and only if T transforms any bounded monotone increasing sequence in $C_0(S)$ into a sequence converging weakly in Y. If $\{x_n\}$ is a bounded monotone increasing sequence in $C_0(S)$, it suffices to show $x = \sigma(M(S)', M(S))\text{-}\lim_n x_n$ is in $N(C_0(S))$ (Theorem 1.1 of [2]). Since then T being a uc operator would imply $T''(x) \in JY$ and, hence, $T(x_n)$ converges weakly to some $y \in Y$. Define $z_1 = x_1$, $z_2 = x_2 - x_1, \ldots, z_n = x_n - x_{n-1}, \ldots$. Then Σz_n is a series in $C_0(S)$.

If $\mu \in M(S)$, then $\mu(x_n - \sum_{i=1}^{n} z_i) = \mu(0) = 0$; hence, $\{x_n - \sum_{i=1}^{n} z_i\}$ converges weakly to 0. Since x_n is a weak Cauchy sequence, $\lim_n \mu(x_n) < \infty$ for each $\mu \in M(S)$. To show Σz_n is a wuc series, it suffices to only consider positive Radon measures, so let μ be an arbitrary positive Radon measure. Since $x_n(s) - x_{n-1}(s) \geq 0$ for all $s \in S$, $|\mu(z_n)| = \mu(z_n)$ and, thus,

$$\lim_n \sum_{i=1}^{n} |\mu(z_i)| = \lim_n \sum_{i=1}^{n} \mu(z_i) = \lim_n \sum_{i=1}^{n} \int_S (x_i - x_{i-1}) d\mu = \lim_n \mu(x_n) < \infty .$$

Hence Σz_n is indeed a wuc series. Now since $\{x_n - \sum_{i=1}^{n} z_i\}$

converges weakly to 0, the weak limit point of $\{x_n\}$ is in $N(C_0(S))$.

We now generalize Theorem 1, for if S is a compact Hausdorff space, then $C(S) = A(K)$ where K is the compact convex set of probability measures on S in the weak* topology [3].

<u>Theorem 2</u>. If the set of extreme points of K is a countable union of compact sets, then A(K) has property V.

<u>Proof.</u> By [2], it suffices to show that any equicontinuous, convex, balanced, and $\sigma(A(K)', N(A(K)))$-compact set D in $A(K)'$ is also $\sigma(A(K)', A(K)'')$-compact. If $\{w_d\}$ is a net in D, then there is a subnet $\{u_a\}$ that converges to some w in D. Let the elements of B be the point-wise limits of series of the form $\Sigma |f_n(x)| < \infty$ for $x \in K$, the f_n's being continuous functions on K. For each bounded Borel function f on K let $(Pf)(x) = \int f\, dw_x$, where for each $x \in K$, w_x is the unique maximal probability measure which represents x. Then for f in B, $P(f)$ is in $N(A(K))$ and since K is maximally supported and $Pf = f$ on the extreme points of K, $\int f\, du = \int Pf\, du$ for each $u \in A(K)'$. Thus $u_a(Pf) \rightarrow w(Pf)$ implies $u_a(f) \rightarrow w(f)$ and $\{u_a\}$ converges to w relative to the $\sigma(C(K)', B)$ topology. Since C(K) has property V, D is compact in the $\sigma(C(K)', C(K)'')$ topology and hence in the $\sigma(A(K)', A(K)'')$ topology.

Recently the Radon-Nikodym property (RNP) has been studied for Banach spaces. (Every separable subspace of X is isomorphic to a subspace of a separable dual - is one among several equivalents for RNP.) It is natural to ask if there is a relation between property V and RNP. By using results of [4] and [5], we have the following.

Proposition 3. Let X be a closed subspace of a Banach space with an unconditional basis. Then X' has RNP if and only if X has property V.

Corollary 4. Let X be a closed subspace of a Banach space with an unconditional basis. If X is a dual space and X' has RNP, then X is reflexive.

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