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A TERNARY VARIETY GENERATED BY LATTICES

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Abstract: The class of all subreducts of lattices, with respect to the ternary lattice-polynomial $s(x,y,z) = x \wedge (y \vee z)$, is a 4-based variety. Within its lattice of subvarieties, the subvariety of distributive supremum algebras is atomic and needs 4 variables in any equational description.

Key words: Subreduct, lattice, nearlattice, distributivity.

Classification: 03C05, 06A12, 06D99, 08B10

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0. Introduction. Consider the variety of algebras $(A;s)$ of type $\langle 3 \rangle$ that satisfy the following identities

(S1) $s(x,x,y) = x$,

(S2) $s(x,y,y) = s(y,x,x)$,

(S3) $s(x,y,z) = s(x,z,y)$,

(S4) $w \wedge s(x,y,z) = s(w \wedge x,y,z)$,

(S5) $s(x,s(y,v,w),\ s(z,v,w)) \leq s(x,v,w)$,

where $\wedge$ is the derived operation given by

(D1) $x \wedge y = s(x,y,y)$,

and $\leq$ is the derived relation defined by

(D2) $x \leq y$ if and only if $x = x \wedge y$.

Then, we show that an algebra is in this variety if and only if it is a subalgebra of a reduct $(L;s)$, where $(L;\wedge,\vee)$ is a lattice and $s(x,y,z) = x \wedge (y \vee z)$. The variety generated a single such lattice-reduct $(L;s)$ has already been studied by Baker [1]; he showed that the variety is congruence-4-distributive but not 3-distributive. Of course, his result extends to our larger variety. However, this can be obtained from Hickman [2], as each reduct $(A;j)$ of an algebra $(A;s)$, where j is the derived ternary operation given by

(D3) $\quad j(x,y,z) = s(y, x\wedge y, y\wedge z)$,

is a join algebra. We demonstrate this by showing that each algebra is a nearlattice wherein $j(x,y,z) = (x\wedge y)\vee(y\wedge z)$. Baker's result has already been exploited by Berman [2].

In the presence of (S1) - (S4), (S5) is implied by

(S6) $\quad w\wedge s(x,y,z) = s(x, w\wedge y, w\wedge z)$.

The subvariety defined by the five identities (S1) - (S4) and (S6), consists of subalgebras of reducts of distributive lattices and so is the variety generated by $(2;s)$, where $(2;\wedge,\vee)$ is the two element lattice. This variety was described by Lyndon [8; Theorem 3] in 1951, using nine identities.

1. Subreducts. Before proceeding to the main results, we would like to make some remarks on subreducts, that is subalgebras of reducts, of the algebras in a variety.

Let $\underline{V}$ be a variety of finitary algebras, p be an n-ary $\underline{V}$-polynomial, and $\underline{V}_p$ be the class of all subalgebras of algebras having the form $(A;p)$, where A is the underlying set of a $\underline{V}$-algebra. Then, $\underline{V}_p$ is a class of algebras of type $\langle n\rangle$, but it is not necessarily a variety. For example, if $\underline{V}$ is the variety of Abelian groups and p is the group-product, then $\underline{V}_p$ is the class of all commutative cancellative semigroups and the cancellation law can fail in homomorphic images of such semigroups. In general, $\underline{\underline{V}}_p$ is closed under the formation of products and subalgebras. A sufficient condition to ensure closure under homomorphic images is as follows.

Suppose p is congruence-determining in the sense that the $\underline{\underline{V}}$-congruences on a $\underline{\underline{V}}$-algebra are precisely the $\underline{\underline{V}}_p$-congruences on its $\underline{\underline{V}}_p$-reduct. Also, assume that $\underline{\underline{V}}_p$ enjoys the Congruence Extension Property. Then, it is immediate that $\underline{\underline{V}}_p$ is closed under homomorphic images and so is a variety. In the above case concerning Abelian groups, p is congruence-determining but the Congruence Extension Property fails for commutative cancellative semigroups. An instance where these conditions hold is supplied by the case of $\underline{\underline{V}}$ being the variety of Boolean algebras $\langle A;\wedge,\vee,',0,1\rangle$ and p being the binary polynomial $p(x,y) = x{*}y = x\wedge y'$. Here, $\underline{\underline{V}}_p$ is the variety of implicative BCK-algebras; for an account, see [3] and especially Corollary 1.5, therein. Thus, $\underline{\underline{V}}_p$ has the Congruence Extension Property; in fact, any variety of BCK-algebras has the Congruence Extension Property, [5; Theorem 3.1]. It is not hard to show that $p(x,y) = x{*}y$ is congruence-determining on a Boolean algebra, or alternatively, this can be deduced from [4, Corollary 3.13].

However, each of these two conditions is not necessary for $\underline{\underline{V}}_p$ to be a variety. For instance, when $\underline{\underline{V}}$ is the variety of (distributive) lattices $\langle A;\wedge,\vee\rangle$ and p is given by $p(x,y) = x\wedge y$, $\underline{\underline{V}}_p$ is the variety of semilattices; here, $\underline{\underline{V}}_p$ has the Congruence Extension Property but p is not congruence

-determining. On the other hand, the main result of this paper, as described in Section 0, supplies an example where the Congruence Extension Property fails (because it fails in any variety of non-distributive lattices) and p, that is s, is congruence-determining. Actually, Baker commented on this property of s in [1; Note 1, p. 143]; this can also be seen from Hickman [7; Proposition 2.2], since (D3) of Section 0 gives Hickman's j as a polynomial in terms of s. Another such example is provided by Hickman's join algebras, see [7; Proposition 1.1(iii), Theorem 2.1]. Thus, the main result of this note has a significance with respect to a model-theoretic problem.

2. Main results. A partially ordered set is said to have the upper bound property if any two elements possess a supremum, whenever they have a common upper bound. A nearlattice is a lower semilattice with the upper bound property. In any nearlattice $(A; \wedge, \dot{\vee})$, $j(x,y,z) = (x \wedge y) \vee (y \wedge z)$ is defined on the whole of A. The resulting ternary algebras $(A; j)$ are equationally definable and are called join algebras. Actually, in [7; Theorem 2.1, Proposition 2.4], Hickman showed that the variety of join algebras and their homomorphisms is isomorphic to the category of nearlattices and their nearlattice-homomorphisms; the isomorphism commutes with the forgetful functors to the category of sets. We will give no further details; additional information on nearlattices and their congruences etc. can be found in [4; Section 3]. It should be mentioned, however, that a lower semilattice is a

nearlattice if and only if the new semilattice, that results from the addition of a largest element, is really a lattice. Hence, the variety of join algebras is the variety $\underline{\underline{V}}_p$, when $\underline{\underline{V}}$ is the variety of lattices and p is j.

We make use of the notation of Section 0. A ternary algebra $(A;s)$, satisfying (S1) - (S5), will be called a <u>supremum algebra</u>.

Lemma 2.1. <u>Let</u> $(A;s)$ <u>be a supremum algebra</u>. <u>Then</u>, $(A;\wedge,\leq)$ <u>is a nearlattice</u>. <u>For any</u> $a,b,c\in A$, $s(a,b,c) = a\wedge(b\vee c)$, <u>when</u> $b\vee c$ <u>exists</u>.

<u>Proof</u>. By (S1) and (D1), $a\wedge a = s(a,a,a) = a$. From (S2), $a\wedge b = b\wedge a$. From (S4), $a\wedge(b\wedge c) = a\wedge s(b,c,c) = s(a\wedge b,c,c) = (a\wedge b)\wedge c$. Thus, (D2) says that $(A;\wedge,\leq)$ is a lower semilattice, wherein $\leq$ is the induced partial order.

Suppose $b,c,\leq a$ and let $d = s(a,b,c)$. Due to (S4) and (S1), $b\wedge d = s(b\wedge a,b,c) = s(b,b,c) = b$, i.e. $b\leq d$. Similarly, $c\leq d$, and d is a common upper bound of b and c. Let e be another such bound. Then, $d = s(a,b,c) = s(a,b\wedge e,c\wedge e) = s(a,s(b,e,e),s(c,e,e))\leq s(a,e,e) = a\wedge e\leq e$, by (S5). Hence, $d = b\vee c$.

When $b\vee c$ exists, the above reasoning shows that $b\vee c = s(b\vee c,b,c)$. Hence, $a\wedge(b\vee c) = a\wedge s(b\vee c,b,c) = s(a\wedge(b\vee c,),b,c) = (b\vee c)\wedge s(a,b,c)\leq s(a,b,c)$, due to two applications of (S4). But due to (S5), $s(a,b,c) = s(a,b\wedge(b\vee c),c\wedge(b\vee c)) = s(a,s(b,b\vee c,b\vee c),s(c,b\vee c,b\vee c))\leq s(a,b\vee c,b\vee c) = a\wedge(b\vee c)$. Hence, $s(a,b,c) = a\wedge(b\vee c)$.

Notice that the lemma implies that j, as defined by (D3), is given by $j(x,y,z) = (x\wedge y)\vee(y\wedge z)$ on the underlying nearlattice of a supremum algebra. Hence, the reduct $(A;j)$ of a supremum algebra $(A;s)$ is a join algebra, and so, by Hickman [7], we obtain

Corollary 2.2. <u>The variety of supremum algebras is congruence-4-distributive, but not congruence-3-distributive.</u>

In order to establish the characterization of supremum algebras as subreducts, we need to introduce the appropriate ideal-theoretic concepts. An <u>s-ideal</u> of a supremum algebra $(A;s)$ is a non-empty subset K such that, for any $a \in A$ and $k_1, k_2 \in K$, $s(a,k_1,k_2) \in K$, When $a \leq k$ and $k \in K$, $a = s(a,k,k)$, and so $a \in K$. If $k_1, k_2 \in K$ and $k_1 \vee k_2$ exists, then Lemma 2.1 implies $k_1 \vee k_2 = s(k_1 \vee k_2, k_1, k_2)$ and so $k_1 \vee k_2 \in K$. Thus, an s-ideal is a <u>nearlattice-ideal</u> of the underlying nearlattice. Nevertheless, the true nature of s-ideals is a mystery to us, even though we can exploit them.

When K_1 and K_2 are s-ideals of a supremum algebra $(A;s)$, there are $k_1 \in K$ and $k_2 \in K$ and so, $k_1 \wedge k_2 = s(k_1 \wedge k_2, k_1, k_2) =$ $= s(k_1 \wedge k_2, k_2, k_2)$, says that $k_1 \wedge k_2$ is in the set-intersection $K_1 \cap K_2$. It follows that $K_1 \cap K_2$ is an s-ideal. Also, let $T_0 = \{a \in A : a = s(a,k_1,k_2),\ k_1 \in K,\ k_2 \in K_2\}$ and $T_{n+1} = \{a \in A :$ $: a = s(a,t_1,t_2),\ t_1, t_2 \in T_n\}$ for $n \geq 1$. Then, we have inductively defined the sequence $K_1, K_2 \subseteq T_0 \subseteq T_1 \subseteq \ldots \subseteq T_n \subseteq T_{n+1} \subseteq \ldots$. Moreover, it is not hard to show that $\bigcup\{T_n : n \geq 0\}$ is the smallest s-ideal containing both K_1 and K_2. Hence, when ordered by set-inclusion, the s-ideals of $(A;s)$ form a lattice, wherein the infimum and supremum of s-ideals K_1 and K_2 are given by $K_1 \cap K_2$ and $K_1 \vee K_2 = \bigcup\{T_n : n \geq 0\}$, respectively. Also, it is not hard to see that for any $b \in A$, $(b] = \{a \in A : a \leq b\}$

is the s-ideal generated by b. In general, the s-ideal generated by a non-empty subset B of A is $\cup\{S_n : n \geq 0\}$, where $S_0 = \{a \in A : a = s(a,b_1,b_2), b_1,b_2 \in B\}$, $S_{n+1} = \{a \in A : a = s(a,r_1,r_2), r_1,r_2 \in S_n\}$, for $n \geq 1$.

Lemma 2.3. <u>Let</u> $(A;s)$ <u>be a supremum algebra and</u> $a,b,c \in A$. <u>Then, in the lattice of</u> s<u>-ideals</u> $(b] \vee (c] = \{d \in A : d = s(d,b,c)\}$, <u>and consequently</u> $(a] \cap ((b] \vee (c]) = (s(a,b,c)]$.

<u>Proof</u>. Let $T = \{d \in A : d = s(d,b,c)\}$. Then, $b,c \in T$, by (S1) and (S3). Also, if $d_1 \leq d$ and $d = s(d,b,c)$ then $d_1 = (d_1,b,c)$, due to (S4). In other words, T is hereditary. Now, if $t_1,t_2 \in T$ and $e \in A$, then, because of (S5) and (S4), $s(e,t_1,t_2) = s(e,s(t_1,b,c), s(t_2,b,c)) \leq s(e,b,c)$, and $s(e,b,c) = s(e,b,c) \wedge s(e,b,c) = s(s(e,b,c) \wedge e,b,c) = s(s(e,b,c),b,c)$. Hence, $s(e,b,c) \in T$, and so $s(e,t_1,t_2) \in T$, too. Thus, T is an s-ideal containing b and c. It immediately follows that $T = (b] \vee (c]$.

The remaining assertion follows quickly from (S4), and its consequence that $s(a,b,c) \leq a$.

Theorem 2.4. <u>Let</u> $\underline{\underline{L}}$ <u>be the variety of all lattices and</u> s <u>be the ternary lattice-polynomial</u> $s(x,y,z) = x \wedge (y \vee z)$. <u>Then,</u> $\underline{\underline{L}}_s$ <u>is the variety of supremum algebras</u>.

<u>Proof</u>. It is easily verified that each algebra in $\underline{\underline{L}}_s$ satisfies (S1) - (S5). On the other hand, a supremum algebra $(A;s)$ is in $\underline{\underline{L}}_s$, as the map $a \longrightarrow (a]$ is a supremum algebra-embedding of $(A;s)$ into the s-reduct of its lattice of s-ideals.

We now turn to distributivity. A nearlattice is <u>distributive</u>, when the infimum distributes over existent finite suprema. This is equivalent to the associated join algebra satisfying the identity $w \wedge j(x,y,z) = j(w \wedge x, y, w \wedge z)$, where $x \wedge y = j(x,y,x)$; see [7; Theorem 3.3]. More importantly, a nearlattice is distributive if and only if either each initial segment is a distributive sublattice or the lattice of nearlattice-ideals is distributive or the finitely generated nearlattice-ideals form a distributive lattice, when ordered by set-inclusion. The equivalence for these last two conditions is contained in the proof of [6; Theorem 1.2].

Theorem 2.5. _The following conditions on a ternary algebra_ $(A;s)$ _are equivalent._

(i) $(A;s)$ _is a supremum algebra satisfying the identity_

(S7) $s(x,y,z) = s(x, x \wedge y, x \wedge z)$.

(ii) $(A;s)$ _is a supremum algebra, whose lattice of s-ideals is distributive._

(iii) $(A;s)$ _satisfies the identities_ (S1) - (S4) _and_ (S6), _where_ $\wedge$ _is defined by_ (D1).

Proof. (i) $\Rightarrow$ (ii). It follows from (i) and Lemma 2.1 that each initial segment of the underlying nearlattice is a distributive lattice. It also follows from (i) and Lemma 2.1 that each nearlattice-ideal is an s-ideal. Thus, the two concepts of ideal coincide here. As the nearlattice is distributive, (ii) follows.

(ii) $\Rightarrow$ (iii) holds because a distributive lattice satisfies (S6).

(iii) $\Rightarrow$ (i). Reasoning along much the same lines as in the proof of Lemma 2.1, we see that (iii) implies that $(A;\wedge,\leq)$ is a nearlattice. Also, $s(a,b,c) = b \vee c$, whenever a is

an upper bound of both b and c. Due to (S6), $s(x,y,z) =$ $= s(x\wedge x,y,z) = x\wedge s(x,y,z) = s(x,x\wedge y,x\wedge z)$. Hence, (S7) holds and $s(x,y,z) = (x\wedge y)\vee(x\wedge z)$. Then, (S6) says that the infimum distributes over such a supremum. Using these observations, it is possible to express the left side of (S5) as a supremum of infima, and so establish (S5).

Corollary 2.6. _Let_ $\underline{\underline{D}}$ _be the variety of distributive lattices and_ s _be the ternary_ $\underline{\underline{D}}$_-polynomial_ $s(x,y,z) = x\wedge(y\vee z) = (x\wedge y)\vee(x\wedge z)$. _Then,_ $\underline{\underline{D}}_s$ _is the variety of all algebras satisfying the conditions of Theorem 2.5._

Call the algebras of Theorem 2.5, _distributive_ supremum algebras. The above results show that Hickman's distributive join algebras and distributive supremum algebras are definitionally equivalent. And, perhaps, it would be more natural to describe join algebras by $(x\wedge y)\vee(x\wedge z)$ instead of j. Corollary 2.6 ensures our remarks at the end of Section 0. Hickman's equational base for the variety of distributive supremum algebras has 9 identities, as does Lyndon's. Ours has 5 identitities. However, both the variety of supremum algebras, and its subvariety of distributive algebras, can be defined by using at most 4 identities. This is because of Corollary 2.2, and Padmanabhan and Quackenbush [9; Theorem 1].

A related question is that of the minimum number of variables needed in an equational base for one of these varieties.

Theorem 2.7. _An equational base for either supremum algebras or distributive supremum algebras need at least_ 4 _va-_

riables. *And* 4 *is sufficient for distributive supremum algebras*.

Proof. Suppose 3 variables are sufficient for either of the varieties. Because of Theorem 2.5 and (S7), 3 variables suffice for distributive supremum algebras. Hence 3 variables suffice to equationally describe distributive join algebras. Now consider the 5-element modular non-distributive lattice. The associated join algebra is not distributive, yet all of its 3-generated join subalgebras are distributive. This gives the desired contradiction. The remaining assertion follows from Theorem 2.5(iii).

When looking for examples of supremum algebras, it is important to observe that any hereditary subset of a lattice is closed under s, and so becomes a supremum subalgebra. We do not know whether all supremum algebras arise this way. But distributive supremum algebras do! As observed in the proof of Theorem 2.5, the s-ideals and the nearlattice-ideals coincide on a distributive supremum algebra. Moreover, the finitely generated nearlattice-ideals form a lattice when the nearlattice is distributive, c.f. [6; Theorem 1.2]. Then, due to the upper bound property, a distributive supremum algebra is a hereditary subset of its lattice of finitely generated s-ideals.

Another feature of distributive supremum algebras is that they are s-isomorphic if and only if they are order-isomorphic. In general, this is not the case. For let us consider the 4-element nearlattice ., which has 3 atoms a, b, and c, and smallest element 0. It is a hereditary subset of the following 4 lattices L_1,L_2,L_3,L_4. Here, L_1 is the Boolean lattice with a,b, and c as atoms; the associated algebra $A_1 = (A;s)$ is distributive, but not subdirectly irreducible. Then, L_2 is the 5-element modular non-distributive lattice; the algebra $A_2 = (A;s)$ is simple and not distributive. The lattice L_3 has 5 elements; the new elements are d and e, $d = b\vee c$, $e = a\vee d = a\vee b\vee c$; the algebra $A_3 = (A;s)$ is subdirectly irreducible, but not simple. The fourth lattice L_4 has 6 elements; the new elements are d,e,f, $d = a\vee b$, $e = b\vee c$, $f = d\vee e$; the algebra $A_4 = (A;s)$ is also subdirectly irreducible but not simple. Up to s-isomorphism, A_1 - A_4 are the only distinct supremum algebras which contain A as a subnearlattice. Because of congruence-distributivity, A_2, A_3 and A_4 generate distinct varieties which cover the variety of distributive supremum algebras. Thus, to us, the study of the lattice of subvarieties of supremum algebras seems hopelessly difficult.

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