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ON TIGHTNESS IN CHAIN-NET SPACES

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Abstract: The question was raised in [2] as to whether every chain-net space with countable tightness is sequential (no separation axioms were imposed). In this paper we construct a number of examples to show that the answer to the above question is no, both in the class of T_2 chain-net spaces and in the class of chain-net spaces in which convergent chains have unique limits (here called T_c-spaces). We also prove that a T_2 chain-net space with countable spread has countable tightness, but need not in general be sequential.

Key words and phrases: Tightness, chain-net space, Fréchet chain-net space, net character, sequential space, spread.

Classification: Primary 54D99, Secondary 54A25

For any cardinal κ , a κ-sequence or chain-net of length κ in a topological space X is a function from κ into X. A space X is said to be a chain-net space if the topological closure of any subset A of X may be obtained by iterating the chain-net closure of A, this latter being obtained by adjoining to A all limits of chain-nets in A. If for each $A \subset X$, the topological closure of A is equal to the chain-net closure of A, then X is said to be a Fréchet chain-

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net space. Chain-net spaces and Fréchet chain-net spaces are called pseudo-radial and radial spaces respectively in [2] and [3].

The notation we use follows [5], but we define two cardinal functions not considered here. If X is a topological space, we define *the net character of* X, $\sigma(X)$, to be the least cardinal λ such that the iteration of the λ-net closure operator yields the topological closure operator. The λ-net closure of a set A is obtained by adjoining to A all limits of nets in A whose directed sets are of cardinality no greater than λ. The cardinal function σ was introduced in [7]. If X is a chain-net space then we define analogously the *chain-net character of* X, $\sigma_c(X)$, by replacing λ-net with λ-sequence in the above definition. If t(X) denotes the tightness of X, and if X is a chain-net space, then

$$t(X) \leq \sigma(X) \leq \sigma_c(X) \leq \exp t(X).$$

It is easy to show that if X is a Fréchet chain-net space then $t(X) = \sigma(X) = \sigma_c(X)$. Furthermore, it was shown in [8], (and independently in [3]), that in this same class of spaces $t(X) \leq s(X)$ (the spread of X). Theorem 1 extends this result to chain-net spaces (it was proved for compact T_2 spaces in [1]). We then construct some examples to show that if X is a chain-net space, then in general $t(X) \neq \sigma(X)$; we need extra axioms to obtain a T_2 example. A space is said to be a T_c-space if every convergent chain-net has a unique limit. Such spaces are clearly T_1 but not in general T_2 (see for instance [4]). All the examples we construct are (at least) T_c-spaces.

<u>Theorem 1</u>. If X is a T_2 chain-net space then $t(X) \leq s(X)$.

Proof. Suppose that for a cardinal λ there is a non-closed subset A of X with the property that if $B \subset A$ and $|B| < \lambda$, then $cl_X B \subset A$. Since A is not closed, there is a μ-sequence ($\mu \geq \lambda$) S in A converging to $p \notin A$. We will define by recursion a relatively discrete subset of S of cardinality λ. Choose $x_1 \in S$ and disjoint open sets U_1 and V_1 such that $x_1 \in U_1$ and $p \in V_1$. Identifying λ with the first ordinal of cardinality λ, suppose that for each $\alpha < \lambda$, we have chosen $x_\beta \in S$ and disjoint open sets U_β and V_β for all $\beta < \alpha$ and satisfying the following conditions:

a) $x_\beta \in U_\beta$ and $p \in V_\beta$.

b) $x_\beta \in (S \cap D_\beta) - \bigcup\{U_\gamma : \gamma < \beta\}$ where $D_\beta = X - cl_X(\{x_\gamma : \gamma < \beta\})$;

then we choose x_α, U_α and V_α as follows:

Since $\alpha < \lambda$ we have that $|\alpha| < \lambda$ and so $|\{x_\beta : \beta < \alpha\}| = |\alpha| < \lambda$. Thus $p \notin cl_X\{x_\beta : \beta < \alpha\}$ and so $D_\alpha = X - cl_X\{x_\beta : \beta < \alpha\}$ is a neighbourhood of p and so must contain a cofinal segment of S. Furthermore, each V_β (for $\beta < \alpha$) must also contain a cofinal segment of S. Thus since we are assuming that μ is a regular cardinal, it follows that $\bigcap\{V_\beta : \beta < \alpha\}$ contains a cofinal segment of S, which implies that S is eventually out of $\bigcup\{U_\beta : \beta < \alpha\}$. Thus we may choose $x_\alpha \in (S \cap D_\alpha) - \bigcup\{U_\beta : \beta < \alpha\}$, and U_α and V_α disjoint open neighbourhoods of x_α and p respectively. Let $F = \{x_\alpha : \alpha < \lambda\}$. Then $|F| = \lambda$ and F is discrete since

$F \cap U_\alpha \cap D_\alpha = \{x_\alpha\}$. Hence $s(X) \geq \lambda$. The result now follows from the definition of $t(X)$.

The above theorem is false if the T_2 separation axiom is weakened to T_c. To obtain a counterexample it suffices to adjoint a point p to a first countable, hereditarily Lindelöf T_2 space X which is left-separated by ω_1 (see [5]); such a space is constructible in ZFC, for example see [9 , page 26]. Neighourhoods of p are of the form $U \cup \{p\}$, where U is open and cocountable in X. The resulting space is a Fréchet chain-net T_c space with countable spread but uncountable tightness. Example 1 shows that a T_2 chain-net space with countable spread need not be sequential. (Compare [3, Theorem 6] or [8, Prop. 5.3] where it is shown that a Fréchet chain-net space of countable spread is Fréchet-Urysohn.)

We now construct three non-sequential chain-net spaces which have countable tightness, thus answering negatively a question of Arhangel´skij [2, question 3), page 43]. The first example is T_2, the others being T_c. The continuum hypothesis is required in the construction of example 1, while example 2 is in ZFC. In both of these examples $\sigma(X) = \aleph_1$. The third example requires MA + ¬ CH for its construction and here $\sigma_c(X) = \exp t(X)$. We do not know whether a T_2 chain-net space with $\sigma_c = \exp t$ can be constructed under MA + ¬ CH.

1. Let X be a first countable, locally countable, regular S-space such as the one constructed using CH in [6, § 1], and let $Y = X \cup \{p\}$ ($p \notin X$), where open neighbourhoods of p are complements of closed countable subsets of X (toge-

ther with the point p). Y is T_2 since X is T_3 and locally countable; Y has countable tightness since X is hereditarily separable; Y is not sequential since clearly no sequence in X can converge to p. However, if $A \subset X$ is such that $p \in cl_Y A$, then $cl_X A$ is obtainable from A by sequence (since X is first countable); furthermore, $cl_X A$ must be uncountable and so any minimal well-ordering of it will converge to p. Hence Y is a chain-net space.

2. If we replace the space X in example 1 by the real line with its usual topology, and adjoin the point p as before, then the resulting space will be a T_c chain net space with countable tightness which is not sequential.

3. Let μ be a minimal cardinal number such that there is a family $\{C_\alpha : \alpha < \mu\}$ of nowhere dense subsets of R whose union is not of first category. Clearly μ is an uncountable regular cardinal.

For $\alpha < \mu$, choose a countable family $\mathcal{T}_\alpha$ of closed nowhere dense subsets of R with $\bigcup \mathcal{T}_\alpha \supseteq \bigcup\{C_\beta : \beta < \alpha\}$. Choose, if possible, a point $x_\alpha \in C_\alpha - \bigcup\{\bigcup \mathcal{T}_\beta : \beta \leq \alpha\}$, let $A = \{x_\alpha : \alpha < \mu\}$. According to the choice of the family $\{C_\alpha : \alpha < \mu\}$, we have $|A| = \mu$.

If $X \subseteq A$ has an uncountable regular cardinality $\kappa < \mu$, then for some $\alpha < \mu$, $X \subseteq \bigcup\{C_\beta : \beta < \alpha\} \subseteq \bigcup \mathcal{T}_\alpha$. Since κ is regular uncountable, $|X \cap T| = \kappa$ for some $T \in \mathcal{T}_\alpha$, for $\mathcal{T}_\alpha$ is countable. But then $cl_A(X \cap T) = cl_R(X \cap T) \cap A \subseteq T \cap A \subseteq$ $\subseteq \{x_\beta : \beta < \alpha\}$.

Thus $cl_A(X \cap T)$ has cardinality less than μ.

Now let $Y = A \cup \{p\}$ $(p \notin R)$, where A has the relative topology from R and open neighbourhoods of p are complements of closed subsets of A of cardinality less than μ (together with p).

It is easy to check that Y is a T_c chain-net space with countable tightness. Furthermore, if $\kappa < \mu$ and S is a κ-sequence in A, then by the above construction, there is a subset $W \subseteq S$ of cardinality κ (and hence cofinal in S) such that $cl_A W$ has cardinality less than μ. Thus S is frequently outside of a neighourhood of p and so does not converge to p. Thus $\sigma_c(Y) = \mu$.

It is well-known that $\mu = 2^{\aleph_0}$ under MA. Thus assuming MA + $\neg$ CH, we have $\sigma_c(Y) = \exp t(Y) = 2^{\aleph_0} > \aleph_1$. We do not know the value of $\sigma(Y)$ in this case.

4. Our last example exhibits consistently a T_2 ccc separable chain-net space X of cardinality greater than $2^{\aleph_0}$. Notice that Archangel'skij proved that if X is a Fréchet chain-net T_2 space, then $|X| \leq 2^{d(X)}$, $|X| \leq d(X)^{c(X)}$ [3].

Let N be a set of natural numbers; for $A, B \subseteq N$ denote $A \overset{*}{\supset} B$ iff $|B - A| < \aleph_0$, $|A - B| = \aleph_0$. A tower $\mathcal{T}$ of length ν is a family $\mathcal{T} = \{T_\alpha : \alpha < \nu\} \subseteq \mathcal{P}(N)$ such that $T_\alpha \overset{*}{\supset} T_\beta$ whenever $\alpha < \beta < \nu$. A tower $\mathcal{T}$ is nowhere dense if for each $A \in [N]^\omega$ there is some $T_\alpha \in \mathcal{T}$ with $|A - T_\alpha| = \aleph_0$. Define

$\vartheta = \min \{|\mathcal{T}| : \mathcal{T}$ is a nowhere dense tower$\}$.

Clearly ϑ is an uncountable regular cardinal.

<u>Claim</u>: There exists a family $\{T_f, A_f : \alpha < \vartheta,\ f \in {}^\alpha 2\}$ of subsets of N satisfying the following:

If $\alpha < \beta < \vartheta$, $f \in {}^{\alpha}2$, $g \in {}^{\beta}2$, then $T_f \supset^* A_g \cup T_g$ provided that $f \subseteq g$, or $|T_f \cap T_g| < \aleph_0$ provided that for some $\gamma \in \operatorname{dom} f \cap \operatorname{dom} g$, $f(\gamma) \neq g(\gamma)$.

Indeed, define by induction $T_\emptyset = \{N\}$, $A_\emptyset = \emptyset$. Let $\xi < \vartheta$ and suppose that T_f, A_f have been defined for all $\alpha < \xi$, $f \in {}^{\alpha}2$. If $\xi = \eta + 1$, $f \in {}^{\eta}2$, choose four infinite disjoint subsets of T_f and denote them $T_{f\cup(\eta,0)}$, $T_{f\cup(\eta,1)}$, $A_{f\cup(\eta,0)}$, $A_{f\cup(\eta,1)}$ respectively. If ξ is a limit ordinal, $f \in {}^{\xi}2$, then by the assumption a family $\mathcal{T}_f = \{T_{f\restriction\alpha} : \alpha < \xi\}$ is a tower which cannot be nowhere dense for $\xi < \vartheta$. Hence there is an infinite set $B_f \subseteq N$ with $B_f \subset^* T_{f\restriction\alpha}$ for each $\alpha < \xi$. Choose two disjoint infinite subsets of B_f and denote them T_f and A_f.

Having proved the claim we shall construct the space X. The underlying set of X is

$$N \cup \bigcup \{{}^{\xi}2 : 0 < \xi < \vartheta\} \cup {}^{\vartheta}2.$$

The topology is defined as follows:

(a) Each point of N is isolated.

(b) If $0 < \xi < \vartheta$, $f \in {}^{\xi}2$, then the basic neighbourhood $O(f,K)$ of f is the set $\{f\} \cup (A_f - K)$, where K runs over all finite subsets of N.

(c) If $\varphi \in {}^{\vartheta}2$, then the basic neighbourhood of φ depends on $\alpha < \vartheta$ and on a choice of neighbourhoods of $\varphi \restriction \xi$:

$$U(\varphi, \alpha, \{O_{\varphi\restriction\xi} : \alpha < \xi < \vartheta\}) = \{\varphi\} \cup \bigcup \{O_{\varphi\restriction\xi} : \alpha < \xi < \vartheta\},$$

where each $O_{\varphi\restriction\xi}$ is a neighbourhood of $\varphi \restriction \xi$.

The space X is obviously ccc and separable, each $f \in {}^{\alpha}2$ can be reached by a convergent sequence, each $f \in {}^{\vartheta}2$ can be reached by a convergent net of length ϑ. So X is a chain-net space. Clearly $|X| = 2^{\vartheta}$.

We need to show that X is Hausdorff. To this end, let $\varphi, \psi \in {}^{\vartheta}2$, $\varphi \neq \psi$. There is some $\alpha < \vartheta$ such that $\varphi \restriction \alpha \neq \psi \restriction \alpha$, hence $T_{\varphi \restriction \alpha} \cap T_{\psi \restriction \alpha}$ is finite. Let $K \subseteq N$ be a finite set such that $(T_{\varphi \restriction \alpha} - K) \cap T_{\psi \restriction \alpha} = \emptyset$. For $\xi > \alpha$, $A_{\varphi \restriction \xi} \subset^{*} T_{\varphi \restriction \alpha}$ and $A_{\psi \restriction \xi} \subset^{*} T_{\psi \restriction \alpha}$, so there are finite sets K_{ξ}, $L_{\xi} \subseteq N$ with $A_{\varphi \restriction \xi} - K_{\xi} \subseteq T_{\varphi \restriction \alpha} - K$, $A_{\psi \restriction \xi} - L_{\xi} \subseteq T_{\psi \restriction \alpha}$. Consequently the neighbourhoods

$$U(\varphi, \alpha, \{O(\varphi \restriction \xi, K_{\xi}) : \alpha < \xi < \vartheta\})$$

and $U(\psi, \alpha, \{O(\psi \restriction \xi, L_{\xi}) : \alpha < \xi < \vartheta\})$ are disjoint.

The proof of separating of other pairs of points from X is similar and may be left to the reader.

It remains to notice that CH or P(C) implies $\vartheta = 2^{\aleph_0}$. Since $|X| = 2^{\vartheta}$, it is consistent with usual axiomes of ZFC that $|X| > 2^{\aleph_0}$.

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