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Stanislav Jendroľ; Zdeněk Ryjáček
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COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE

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3-POLYTOPES OF CONSTANT TOLERANCE OF EDGES

Stanislav JENDROĽ, Zdeněk RYJÁČEK

Abstract: We prove necessary and sufficient conditions for the existence of planar graphs (and especially for 3-polytopal graphs) of constant tolerance of edges, i.e. graphs each edge of which has the same absolute value of the difference of valencies of its vertices.

Key words: 3-polytope, planar graph, plane graph, degree set.

Classification: 05C10
52A25

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1. Introduction. By the tolerance of an edge AB of a 3-polytope P is meant the absolute value of the difference of valencies of the vertices A, B. All 3-polytopes of regular graphs have obviously constant tolerances of edges. In the present note, 3-polytopes of constant tolerance of edges, i.e. the 3-polytopes each edge of which has the same tolerance, will be investigated and necessary and sufficient conditions for their existence will be given.

A theorem of Steinitz [3] states that a graph is the graph of a 3-polytope if and only if it is planar and 3-connected. Defining analogously the notion of a graph of constant tolerance of edges (briefly ct-graph in the sequel), one can easily observe that the study of the existence of planar 3-connected ct-graphs is sufficient.

Let $G = (V,E)$ be a connected ct-graph with the tolerance t, $t \geq 0$. Then the following assertions can be easily proved:

i) If $t = 0$, then G is regular.

ii) If $t > 0$, then G is bipartite.

iii) Let $D_G(V) = \{m;\ m = \deg_G(v) \text{ for some } v \in V\}$ be the degree set of G; $k = |D_G(V)|$ its cardinality.

Denote by $\delta = \delta(G)$ the minimum degree of G. Then

(1) $\quad D_G(V) = \{n;\ n = \delta + it,\ i = 0,1,\dots,k-1\}$

Conversely, the question arises whether, given integers $t \geq 0$, $\delta \geq 1$, $k \geq 1$, there exists a ct-graph G with degree set of the type (1). This problem was solved by Acharya and Vartak [1] without assumption of the planarity of G. Our aim is to study it for the case of planar graphs and especially for graphs of 3-polytopes.

2. Let us formulate now the main result of this paper.

<u>Theorem 1</u>. Let be given integers $t \geq 0$, $\delta \geq 1$, $k \geq 1$. Planar ct-graph with the degree set of the type (1) exists only in the following cases:

i) $t = 0$, $1 \leq \delta \leq 5$, $k = 1$

ii) $t \geq 1$, $1 \leq \delta \leq 2$, $k \geq 2$

iii) $t = 1$, $\delta = 3$, $k \geq 2$

iv) $t = 2$, $\delta = 3$, $k \geq 2$.

Moreover, in each of these cases there exists also a δ-connected ct-graph with the same degree set.

From Theorem 1 one can easily obtain by Steinitz' theorem [2] the following

Theorem 2. Let be given integers $t \geq 0$, $\delta \geq 1$, $k \geq 1$. 3-polytope of constant tolerance of edges with the degree set of the type (1) exists only in the following cases:

i) $t = 0$, $3 \leq \delta \leq 5$, $k = 1$

ii) $1 \leq t \leq 2$, $\delta = 3$, $k \geq 3$.

3. Proof of Theorem 1. We shall first prove that the above conditions are sufficient.

Case i): Consider graphs of K_2, K_3, tetrahedron, octahedron and icosahedron.

Case ii): Let be given integers $t \geq 1$, $\delta \geq 1$, $k \geq 2$. Consider a plane tree (Graphtheoretical terms being used in the sense of [2].) with the following properties:

a) its degree set is $\{1\} \cup \{n;\ n = \delta + it,\ i = 1,\ldots,k-1\}$;

b) each of its 1-valent vertices is adjacent to a $(\delta + t)$-valent vertex;

c) tolerance of each its edge non-incident with any 1-valent vertex is equal to t.

For any fixed t, δ, k one can easily verify the existence of such a tree; let us choose arbitrarily one of them and denote it by $T_{t,\delta,k}$. Further, denote by $v_1 = v_1(T_{t,\delta,k})$ the number of its 1-valent vertices.

Now, for $t \geq 1$, $\delta = 1$, $k \geq 2$ the graph $T_{t,1,k}$ and for $t \geq 1$, $\delta = 2$, $k \geq 2$ the graph obtained by suitable identifying of the 1-valent vertices of two disjoint copies of $T_{t,2,k}$ gives the example of the graph with properties which are required in Theorem 1.

Case iii): For $k = 2$ see Fig. 1. Let $k \geq 3$. Consider two disjoint copies of $T_{1,4,k-1}$ and a plane graph $K_2 \times C_{2r}$, where $r = v_1(T_{1,4,k-1})$. The plane graph $K_2 \times C_{2r}$ has exactly two 2r-gonal faces. Insert one copy of $T_{1,4,k-1}$ into one of them and suitably identify its 1-valent vertices with vertices of the face. Make the same with the second copy of $T_{1,4,k-1}$ in the second face; in this construction the identification must be made so that none of the vertices obtained by identification can be adjacent in the resulting graph. It can be easily seen that the obtained graph has the required properties. For example, Fig. 2 shows the ct-graph obtained by means of the described construction if $k = 4$, i.e. from two copies of $T_{1,4,3}$ and $K_2 \times C_{48}$.

Case iv): Consider the tree $T_{2,3,k}$ and the plane graph C_{2r} (i.e. the cycle of the length 2r), where $r = v_1(T_{2,3,k})$. Insert $T_{2,3,k}$ into the face of C_{2r} and identify the r 1-valent vertices of $T_{2,3,k}$ with r 2-valent vertices of C_{2r} so that none of the vertices obtained by identification can be adjacent in the resulting graph. It can be easily seen that the obtained graph has exactly r 2-valent vertices - unidentified vertices of C_{2r}. Take two disjoint copies of this graph, insert each of them into one of two 2r-gonal faces of the plane graph $K_2 \times C_{2r}$ and identify the r 2-valent vertices of each copy with r vertices of the correspondent 2r-gonal face. This second identification must be made so that again none of the identified vertices can be adjacent in the resulting graph. It can be easily seen that the obtained graph has the required properties. An example of this graph for $k = 3$ is shown on Fig. 3.

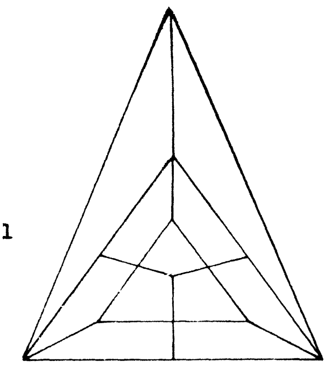

Fig. 1

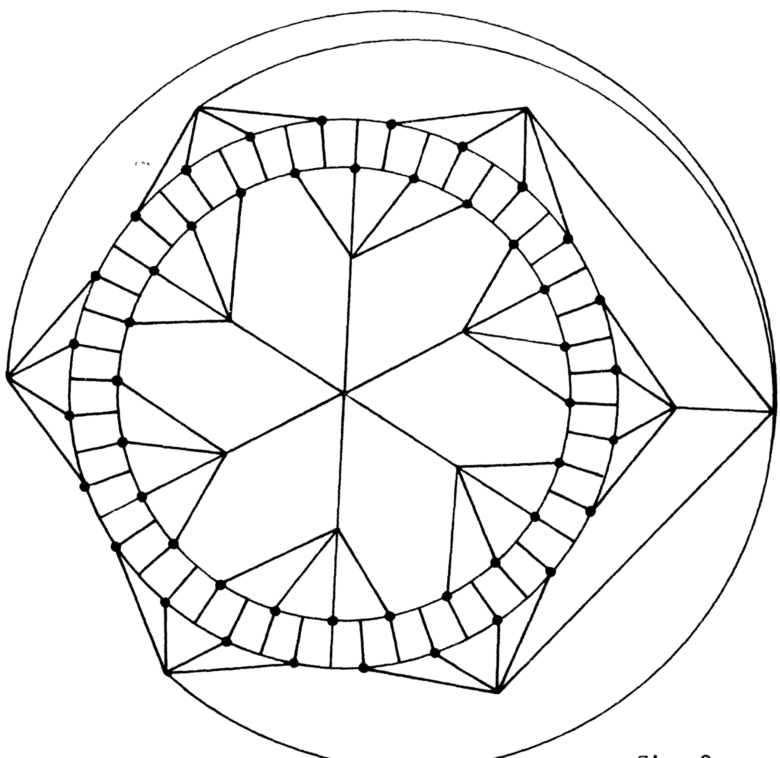

Fig. 2

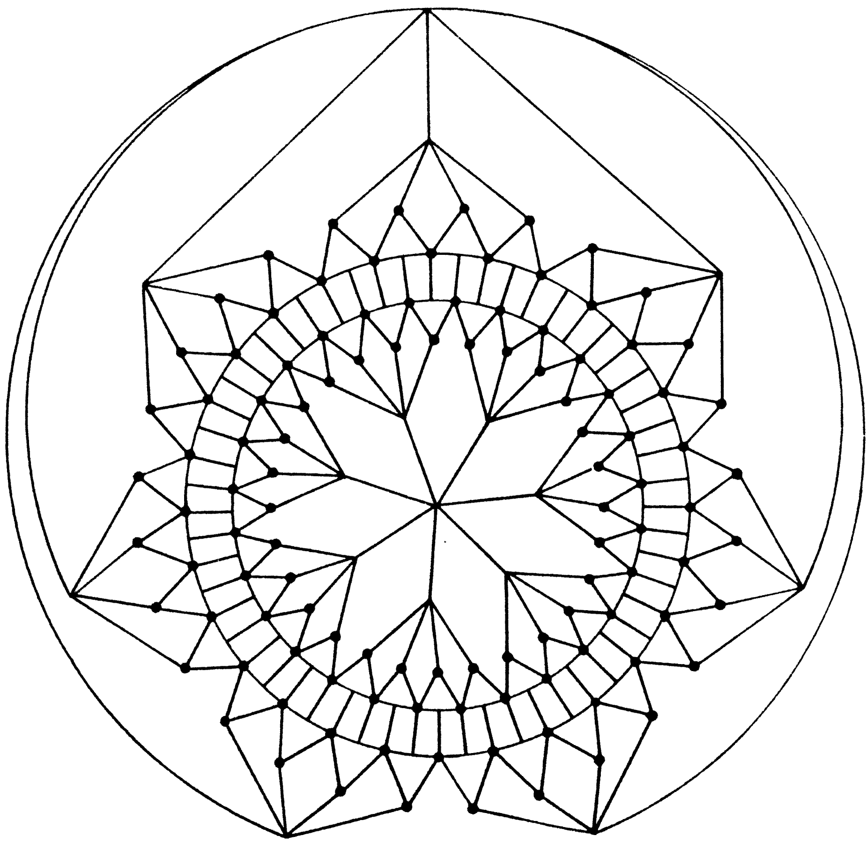

Fig. 3

Now we shall prove the necessity of the conditions.

Case i): Necessity of the condition $\delta \leq 5$ is a well-known consequence of Euler's formula (see [2]), necessity of the condition $k = 1$ is evident.

Cases ii), iii) and iv): From the well-known consequence of Euler's formula for triangle-free graph $G = (V,E)$

$$|E| \leq 2|V| - 4 \tag{2}$$

and from the fact that the ct-graph is bipartite if $t \geq 1$, it follows that

$$2|V| > 2|V| - 4 \geq |E| = \frac{1}{2} \sum_{v \in V} \deg(v) \geq \frac{1}{2} \delta |V|$$

and hence $\delta < 4$. It remains to examine the case $t \geq 3$, $\delta = 3$, $k \geq 2$.

Denote by α_i the number of $(3+it)$-valent vertices of G $(i = 0,1,\ldots,k-1)$. From (2) it follows that

$$\frac{1}{2} \sum_{i=0}^{k-1} (3+it)\alpha_i = 2 \sum_{i=0}^{k-1} \alpha_i - 4$$

and after an adaptation we have

$$\alpha_0 \geq 8 + (t-1)\alpha_1. \tag{3}$$

Denote by e_0 the number of edges which are incident with 3-valent vertices of G. Evidently

$$\alpha_0 = \frac{1}{3} e_0 \text{ and } \alpha_1 \geq \frac{1}{3+t} e_0,$$

from which using (3)

$$0 \geq 8 + \left(\frac{t-1}{t+3} - \frac{1}{3}\right) e_0,$$

therefore $t < 3$, which completes the proof.

4. Remark. Rosenfeld [4] studied 3-polytopes of constant weight i.e. 3-polytopes with constant sum of valencies of vertices on each its edge and proved that such a 3-polytope is either a 3-polytope of regular graph or its degree set is $\{3,4\}$ or $\{3,5\}$. After some considerations it can be shown that all these 3-polytopes are 3-polytopes of constant tolerance of edges.

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Katedra geometrie a algebry
Prírodovedecká fakulta
Univerzity P.J. Šafárika
Komenského 14, 041 54 Košice
Československo

Katedra matematiky VŠSE
Nejedlého sady 14
306 14 Plzeň
Československo

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