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PRERADICALS AND GENERALIZATIONS OF QF-3' MODULES, I

Josef JIRÁSKO

Abstract: QF-3′ modules (i.e. modules Q with $p^{\{Q\}}(E(Q)) = 0$) were studied by various authors (see [2], [10],[12],[14]). Rings with $p^{\{R\}}(X) = 0$ for every finitely generated submodule X of $E(_R R)$ (left QF-3′ rings) were characterized by T. Sumioka [16]. In this paper QF-3″ modules are introduced and are characterized in terms of preradicals. Some results on QF-3′ modules and rings and preradicals connected with QF-3″ modules are obtained.

Key words: F-hereditary preradicals, F-cohereditary preradicals, QF-3″ modules, QF-3″ rings.

Classification: 16A63, 16A36

In the following R stands for an associative ring with unit. The category of all left R-modules will be denoted by R-mod.

A preradical r for R-mod is a subfunctor of the identity functor, i.e. r assigns to each $M \in$ R-mod its submodule r(M) such that $f(r(M)) \subseteq r(N)$ for any $f \in \mathrm{Hom}_R(M,N)$. A preradical r is said to be

- idempotent if $r(r(M)) = r(M)$ for every $M \in$ R-mod,
- a radical if $r(M/r(M)) = 0$ for every $M \in$ R-mod,
- hereditary if $r(N) = N \cap r(M)$ whenever N is a submodule of M, $M \in$ R-mod,
- cohereditary if $r(M/N) = (r(M) + N)/N$ whenever N is a

submodule of M, $M \in R$-mod.

Let r be a preradical. A module M is called

- r-torsion if $r(M) = M$
- r-torsionfree if $r(M) = 0$
- r-splitting if $r(M)$ is a direct summand in M.

The class of all r-torsion (r-torsionfree) modules will be denoted by $\mathcal{T}_r$ ($\mathcal{F}_r$).

We say that a preradical r

- has FGSP if every finitely generated module is r-splitting.

The zero functor will be denoted by zer. For a module Q let us define a radical $p^{\{Q\}}$ by $p^{\{Q\}}(M) = \bigcap \operatorname{Ker} f$ where f runs over all $f \in \operatorname{Hom}_R(M,Q)$, $M \in R$-mod.
Let r, s be preradicals. If $r(M) \subseteq s(M)$ for every $M \in R$-mod then we write $r \leq s$.

The idempotent core $\overline{r}$ of a preradical r is defined by $\overline{r}(M) = \sum K$, where K runs over all r-torsion submodules of M and the radical closure $\tilde{r}$ by $\tilde{r}(M) = \bigcap L$, where L runs over all submodules L of M with M/L r-torsionfree.

If $\{r_i; i \in I\}$ is a family of preradicals then $\bigcap_{i \in I} r_i$ ($\sum_{i \in I} r_i$) is a preradical defined by $(\bigcap_{i \in I} r_i)(M) = \bigcap_{i \in I} r_i(M)$ ($(\sum_{i \in I} r_i)(M) = \sum_{i \in I} r_i(M)$), $M \in R$-mod.

The Jacobsen radical will be denoted by J and the singular preradical by Z.

A module M is called finitely embedded if there is a finitely generated module N such that M is a submodule of N.

The injective hull of a module Q will be denoted by $E(Q)$.
A module M is called nonsingular if $Z(M) = 0$. A module M is called Π-projective if every direct product M^I of copies

of M is projective.

A ring R is called

- left perfect if every left R-module has a projective cover,
- left V-ring if every simple R-module is injective,
- left semiartinian if every nonzero left R-module has nonzero socle.

A preradical r is said to be

- 1-idempotent if $r(M) \in \mathcal{T}_r$ for every finitely generated module M,
- 2-idempotent if $r(M) \in \mathcal{T}_r$ for every finitely embedded module M,
- F-hereditary if $r(A) = A \cap r(B)$ whenever $A \subseteq B$, B finitely generated,
- F_1-hereditary if $r(Q) = 0$ implies $r(X) = 0$ for every finitely generated submodule X of $E(Q)$,
- F-cohereditary if for every module M $r(M) = \sum r(X)$, where X runs over all finitely generated submodules of M.

For a preradical r let us define preradicals $(Fh)(r)$ and $(Fch)(r)$ as follows:

$(Fh)(r)(Q) = r(Q) + \sum (Q \cap r(X))$, where X runs over all finitely generated submodules of $E(Q)$, $Q \in R\text{-mod}$,

$(Fch)(r)(Q) = \sum r(X)$, where X runs over all finitely generated submodules of Q, $Q \in R\text{-mod}$.

Proposition 1.

(i) Every F-hereditary preradical is F_1-hereditary.

(ii) Every F_1-hereditary radical is F-hereditary.

(iii) $(Fh)(r)$ is an F_1-hereditary preradical and $r \leq (Fh)(r)$.

(iv) If $r \leq s$, s F-hereditary then $(Fh)(r) \leq s$.

(v) $(Fh)(r)(Q)$ does not depend on particular choice of $E(Q)$.

(vi) $\widetilde{(Fh)(r)}$ is the least F-hereditary radical containing r.

(vii) $(Fch)(r)$ is an F-cohereditary preradical and $(Fch)(r) \leq r$.

(viii) If $s \leq r$, s F-cohereditary then $s \leq (Fch)(r)$.

(ix) $(Fch)(r)$ is the largest F-cohereditary preradical contained in r.

(x) $(Fch)(r)(Q) = r(Q)$ for every finitely generated module Q.

(xi) $(Fh)(r)(Q) = r(Q)$ for every injective module Q.

(xii) Every hereditary and every cohereditary preradical is F-cohereditary.

(xiii) If $\{r_i; i \in I\}$ is a family of F-hereditary preradicals then $\bigcap_{i \in I} r_i$ is F-hereditary.

(xiv) If r is a preradical then $\bigcap \{s; r \leq s$, s F-hereditary (pre)radical$\}$ is the least F-hereditary (pre)radical containing r.

(xv) If $\{r_i; i \in I\}$ is a family of F-cohereditary preradicals then $\sum_{i \in I} r_i$ is F-cohereditary.

(xvi) If r is a preradical then $\sum \{s; s \leq r$, s F-cohereditary (idempotent) preradical$\}$ is the largest F-cohereditary (idempotent) preradical contained in r.

(xvii) A preradical r is F_1-hereditary if and only if $\tilde{r}$ is F_1-hereditary.

(xviii) If r is F-hereditary then $\tilde{r}$ is so.

(xix) If r is F-hereditary then $\bar{r}$ is so.

__Proof.__ (i). If $r(Q) = 0$, X is a finitely generated submodule of $E(Q)$ and r F-hereditary then $0 = r(Q \cap X) = r(X) \cap \cap (Q \cap X) = Q \cap r(X)$ and hence $r(X) = 0$.

(ii). Let $A \subseteq B$, B finitely generated and r be an $\mathcal{F}_1$-hereditary radical. Consider the following commutative diagram

$$\begin{tikzcd}
(r(B) \cap A)/r(A) \arrow[r, hook] \arrow[d, "f"'] & B/r(A) \arrow[dl, dashed, "g"] \\
E((r(B) \cap A)/r(A)) &
\end{tikzcd}$$

Then $\operatorname{Im} g$ is finitely generated and $(r(B) \cap A)/r(A) \in \mathcal{F}_r$ since r is a radical. Hence $\operatorname{Im} g \in \mathcal{F}_r$ by the assumption. Now $(r(B) \cap A)/r(A) \subseteq g(r(B/r(A))) \subseteq r(\operatorname{Im} g) = 0$ and consequently $r(A) = r(B) \cap A$.

The remaining assertions are clear.

Proposition 2. For a radical r the following are equivalent

(i) r is 1-idempotent (2-idempotent),

(ii) if $0 \to A \to B \to C \to 0$ is exact, B finitely generated (finitely embedded), A, $C \in \mathcal{F}_r$ then $B \in \mathcal{F}_r$.

Proof. (i) implies (ii). It is easy since for an 1-idempotent (2-idempotent) radical and finitely generated (embedded) module F $F \in \mathcal{F}_r$ if and only if $\operatorname{Hom}_R(T,F) = 0$ for every $T \in \mathcal{T}_r$.

(ii) implies (i). Consider the following exact sequence $0 \to r(B)/r(r(B)) \hookrightarrow B/r(r(B)) \to B/r(B) \to 0$, where B is finitely generated (embedded). Now $B/r(r(B)) \in \mathcal{F}_r$ by (ii) and consequently $r(B) \in \mathcal{T}_r$.

Proposition 3. The following are equivalent for a preradical r

(i) r is F-hereditary,

(ii) $r(A) = A \cap r(B)$ whenever $A \subseteq B$, B finitely embedded,

(iii) if $A \xrightarrow{f} r(B)$ is a monomorphism /A cyclic/ and B is finitely generated (embedded) then $A \in \mathcal{T}_r$,

(iv) a) r is 1-idempotent (2-idempotent)

and

b) whenever $A \subseteq B$, $B \in \mathcal{T}_r$ /A cyclic/ , B finitely embedded then $A \in \mathcal{T}_r$.

<u>Proposition 4</u>. The following are equivalent for a preradical r

(i) r is $\mathcal{F}_1$-hereditary,

(ii) $r(Q) = 0$ implies $r(X) = 0$ for every finitely embedded submodule X of $E(Q)$.

<u>Proposition 5</u>. Let r be a preradical. Then r is $\mathcal{F}$-hereditary if and only if $(\mathcal{F}ch)(r)$ is hereditary.

<u>Proof</u>. Suppose r is $\mathcal{F}$-hereditary and $A \subseteq (\mathcal{F}ch)(r)(B)$. Without loss of generality we can assume A is finitely generated. Hence there are finitely generated submodules X_i, $i \in \{1,2,\dots,n\}$ of B such that $A \subseteq \sum_{i=1}^{n} r(X_i) \subseteq r(\sum_{i=1}^{n} X_i)$ and consequently $A \in \mathcal{T}_{(\mathcal{F}ch)(r)}$ since r is $\mathcal{F}$-hereditary and A is finitely generated.

<u>Corollary 6</u>. An $\mathcal{F}$-cohereditary preradical is $\mathcal{F}$-hereditary if and only if it is hereditary.

<u>Proposition 7</u>. Let r be an $\mathcal{F}$-hereditary radical. Then there is an injective $(\mathcal{F}ch)(r)$-torsionfree module Q such that $r(N) = p^{\{Q\}}(N)$ for every finitely embedded module N.

<u>Proof</u>. By Proposition 5 and [3], Theorem 2.5 there is an injective $(\mathcal{F}ch)(r)$-torsionfree module Q such that $\widetilde{(\mathcal{F}ch)(r)} = p^{\{Q\}}$. Hence $r(N) = p^{\{Q\}}(N)$ for every finitely embedded module N.

Proposition 8. Let r be an $\mathcal{F}$-hereditary preradical (radical) and ω is the set of all left ideals I with $R/I \in \mathcal{T}_r$. Then

(i) ω is a (radical) filter.

If s is the hereditary preradical (radical) corresponding to ω then

(ii) $s(M) = \{m \in M; Rm \in \mathcal{T}_r\}$,

(iii) s is the largest hereditary preradical (radical) contained in r,

(iv) $s = (\mathcal{F}ch)(r)$.

A left R-module Q is called

- QF-3´´ if the radical $p^{\{Q\}}$ is $\mathcal{F}$-hereditary,
- i QF-3´´ if the idempotent radical $\overline{p^{\{Q\}}}$ is $\mathcal{F}$-hereditary.

Proposition 9. Let $Q \in$ R-mod. Then the following are equivalent

(i) Q is QF-3´´,

(ii) $p^{\{Q\}}(X) = 0$ for every finitely generated (embedded) submodule X of E(Q),

(iii) if X is a finitely generated (embedded) submodule of E(Q) then X is isomorphic to a submodule of a direct product of copies of Q,

(iv) $(\mathcal{F}ch)(p^{\{Q\}})$ is hereditary,

(v) $p^{\{Q\}}(X) = p^{\{E(Q)\}}(X)$ for every finitely generated (embedded module X,

(vi) $(\mathcal{F}ch)(p^{\{Q\}}) = p^{\{E(Q)\}}$.

(vii) $(\mathcal{F}ch)(p^{\{Q\}})(E(Q)) = 0$,

(viii) for every finitely generated (embedded) module X $p^{\{E(Q)\}}(X) = 0$ implies $p^{\{Q\}}(X) = 0$,

(ix) a) $\mathrm{Hom}_R(p^{\{Q\}}(X),Q) = 0$ for every finitely generated (embedded) module X

and

b) if $A \subseteq B$, /A cyclic/, B finitely embedded and $\mathrm{Hom}_R(B,Q) = 0$ then $\mathrm{Hom}_R(A,Q) = 0$,

(x) a) If $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is exact, B is finitely generated (embedded), $A \in \mathcal{F}_p\{Q\}$ and $C \in \mathcal{F}_p\{Q\}$ then $B \in \mathcal{F}_p\{Q\}$

and

b) if $A \subseteq B$ /A cyclic/, B finitely embedded and $\mathrm{Hom}_R(B,Q) = 0$ then $\mathrm{Hom}_R(A,Q) = 0$,

(xi) a) if $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is exact, B is finitely generated (embedded), $A \in \mathcal{F}_p\{Q\}$ and $C \in \mathcal{F}_p\{Q\}$ then $B \in \mathcal{F}_p\{Q\}$

and

b) for every finitely embedded module X $\mathrm{Hom}_R(X,E(Q)) = 0$ if and only if $\mathrm{Hom}_R(X,Q) = 0$,

(xii) for every monomorphism $h:A \longrightarrow B$, where B is finitely generated (embedded), for every non-zero homomorphism $f:A \longrightarrow Q$ there are homomorphisms $k:\mathrm{Im}\, f \longrightarrow Q$ and $g:B \longrightarrow Q$ with $0 \neq k \circ f = g \circ h$,

(xiii) for every cyclic module C, finitely generated (embedded) submodule X of $E(C)$ with $h:C \hookrightarrow X$ and every non-zero homomorphism $f:C \longrightarrow Q$ there are homomorphisms $k:\mathrm{Im}\, f \longrightarrow Q$ and $g:X \longrightarrow Q$ such that $0 \neq k \circ f = g \circ h$,

(xiv) if A is a /cyclic/ submodule of a finitely generated (embedded) module B and $\mathrm{Hom}_R(A,Q) \neq 0$ then there is a homomorphism $g:B \longrightarrow Q$ with $g(A) \neq 0$.

<u>Proof</u>. The equivalence of the first eleven conditions follows from Propositions 1 (i),(ii), 2, 3 (iv), 4 and 5.

(ii) implies (xii). Consider the following commutative diagram

$$\begin{tikzcd} A \arrow[r, hook, "h"] \arrow[d, "f"'] & B \arrow[ddl, "p"] \\ Q \arrow[d, hook] & \\ E(Q) & \end{tikzcd}$$

where $f \neq 0$ and B is finitely generated. Then $p^{\{Q\}}(\mathrm{Im}\ p) = 0$ by (ii) and hence $0 \neq f(A) \not\subseteq p^{\{Q\}}(\mathrm{Im}\ p)$. Thus there is a homomorphism $q: \mathrm{Im}\ p \to Q$ with $q(f(A)) \neq 0$. Put $k = q|_{f(A)}$ and $g = q \circ p$. Then $0 \neq k \circ f = g \circ h$.

(xiv) implies (ii). Suppose there is a finitely generated submodule X of E(Q) such that $p^{\{Q\}}(X) \neq 0$. Then $L = p^{\{Q\}}(X) \cap Q \neq 0$. Hence there is a homomorphism $g: X \to Q$ with $g(L) \neq 0$ by (xiv), a contradiction.

The rest is clear.

Proposition 10. Let $Q \in R$-mod. Then the following are equivalent

(i) Q is iQF-3'',

(ii) $\mathrm{Hom}_R(Y,Q) \neq 0$ for every finitely embedded nonzero submodule Y of E(Q),

(iii) $(\mathcal{F}ch)(\overline{p^{\{Q\}}})$ is hereditary,

(iv) $\overline{p^{\{Q\}}}(X) = p^{\{E(Q)\}}(X)$ for every finitely generated (embedded) module X,

(v) $(\mathcal{F}ch)(\overline{p^{\{Q\}}}) = p^{\{E(Q)\}}$,

(vi) $(\mathcal{F}ch)(\overline{p^{\{Q\}}})(E(Q)) = 0$,

(vii) for every finitely generated (embedded) module X $p^{\{E(Q)\}}(X) = 0$ implies $\mathrm{Hom}_R(Y,Q) \neq 0$ whenever $0 \neq Y \subseteq X$,

(viii) if $A \subseteq B$ /A cyclic/, B finitely embedded and

$\mathrm{Hom}_R(B,Q) = 0$ then $\mathrm{Hom}_R(A,Q) = 0$,

(ix) for every finitely embedded module X $\mathrm{Hom}_R(X,E(Q)) = 0$ if and only if $\mathrm{Hom}_R(X,Q) = 0$.

Proof. It follows immediately from Propositions 1(i), (ii), 3(iv), 4 and 5.

Proposition 11. Let $Q \in R$-mod. If $p^{\{Q\}}$ has FGSP then Q is QF-3$''$ if and only if Q is i QF-3$''$.

Proof. With respect to Proposition 1(xix) it suffices to prove the "only if" part. Suppose there is $X \subseteq E(Q)$, X finitely generated and $L = p^{\{Q\}}(X) \neq 0$. Then $\mathrm{Hom}_R(L,Q) \neq 0$ by Proposition 10 since Q is iQF-3$''$. Thus there is a nonzero homomorphism $f: L \longrightarrow Q$ which can be extended to a homomorphism $g: X \longrightarrow Q$, a contradiction.

Proposition 12. Let S be a simple R-module. Then S is QF-3$''$ if and only if it is injective.

Proof. Suppose $0 \neq S$ is simple and QF-3$''$, $0 \neq X \subseteq E(S)$, X finitely generated. Then $p^{\{S\}}(X) = 0$. Hence $0 \neq S \nsubseteq p^{\{S\}}(X)$ and consequently there is a homomorphism $f: X \longrightarrow S$ such that $f(S) \neq 0$. Since $\mathrm{Ker}\, f \cap S = 0$, f is an isomorphism. Thus $X = S$. Hence $S = E(S)$ is injective.

A module Q is said to be an F-cogenerator if $p^{\{Q\}}(N) = 0$ for every finitely generated (embedded) module N.

Remark 13. Let $Q \in R$-mod. Then Q is an F-cogenerator if and only if $(F\mathrm{ch})(p^{\{Q\}}) = \mathrm{zer}$.

Proposition 14. For $Q \in R$-mod the following are equivalent

(i) Q is an F-cogenerator,

(ii) Q is QF-$3''$ and $E(Q)$ is a cogenerator,

(iii) Q is QF-$3''$ and every simple R-module is isomorphic to a submodule of Q.

Proof. (ii) is equivalent to (iii). By [12], Proposition 2.8. (i) is equivalent to (ii). It follows immediately from Proposition 9.

Corollary 15. Let Q be an injective R-module. Then Q is an F-cogenerator if and only if it is a cogenerator.

Proposition 16. Let $Q = \prod_{S \in \mathcal{S}} S$, where $\mathcal{S}$ is the representative set of simple left R-modules. Then the following are equivalent

(i) Q is QF-$3''$,

(ii) J is F-hereditary,

(iii) Q is an F-cogenerator,

(iv) R is a left V-ring.

Proof. (i) is equivalent to (iii). It follows from Proposition 14. The rest is clear since $J = p^{\{Q\}}$.

Proposition 17. The following are equivalent for a faithful module Q:

(i) $(Fch)(p^{\{Q\}}) = Z$,

(ii) Q is QF-$3''$ and $Z(Q) = 0$,

(iii) $\mathcal{F}_{(Fch)(p^{\{Q\}})} = \mathcal{F}_Z$.

Proof. (iii) implies (ii). As it is easy to see $Z(Q) = 0$. If $X \subseteq E(Q)$, X finitely generated then $Z(X) = 0$ and consequently $p^{\{Q\}}(X) = 0$.

(ii) implies (i). $Z(Q) = 0$ implies $Z \leq p^{\{Q\}}$ and hence $Z \leq (Fch)(p^{\{Q\}})$. On the other hand if N is finitely embedded and r-torsion, where $r = p^{\{Q\}}$, $n \in N$ and I is a left ideal with $I \cap (0:n) = 0$. Then the homomorphism $f: I \longrightarrow In$ defined by $f(i) = in$, $i \in I$ is an isomorphism. Now $I \in \mathcal{T}_r$ since r is $\mathcal{F}$-hereditary. Hence $I \subseteq r(R) = 0$ by assumption. Thus $(0:n)$ is essential in R and consequently $Z(N) = N$. Therefore $r(N) \leq Z(N)$ for every finitely embedded module N and we have $(\mathcal{F}ch)(p^{\{Q\}}) = Z$.

The rest is clear.

Corollary 18. Let R be a ring with $Z({}_RR) = 0$. Then the following are equivalent

(i) R is a left QF-3'' ring (i.e. ${}_RR$ is QF-3''),

(ii) $(\mathcal{F}ch)(p^{\{R\}}) = Z$,

(iii) $p^{\{R\}}(X) = 0$ whenever $X \subseteq N$, X finitely generated and N nonsingular.

Proposition 19. For an Π-projective faithful R-module Q the following are equivalent

(i) Q is QF-3''

(ii) for every finitely generated submodule X of $E(Q)$ there is a projective module P_X such that X is isomorphic to a submodule of P_X.

Proof. (i) implies (ii). If X is a finitely generated submodule of $E(Q)$ then $p^{\{Q\}}(X) = 0$ and hence X is isomorphic to a submodule of Q^I for some I, which is projective.

(ii) implies (i). If X is a finitely generated submodule of $E(Q)$ and $X \subseteq P$ for some projective module P then $p^{\{Q\}}(X) \subseteq p^{\{Q\}}(P) = 0$ since Q is faithful.

Corollary 20. Let R be left perfect and right coherent ring. Then the following are equivalent

(i) R is left QF-3'',

(ii) for every finitely generated submodule X of $E({}_RR)$ there is a projective module P_X such that X is isomorphic to a submodule of P_X.

Proposition 21.

(i) Every direct product of QF-3'' R-modules is QF-3''.

(ii) Every direct sum of QF-3'' R-modules is QF-3''.

(iii) Every essential extension of a QF-3'' R-module is QF-3''.

Proof. (i). Let $Q = \prod_{i \in I} Q_i$, where Q_i, $i \in I$ are QF-3'' modules. Then $p^{\{Q\}} = \bigcap_{i \in I} p^{\{Q_i\}}$ is $\mathcal{F}$-hereditary by Proposition 1(xiii). Thus Q is QF-3''.

(ii). It can be made similarly as in (i).

(iii). Obvious.

Proposition 22. Let $A, B \subset$ R-mod. If $p^{\{A\}}(B) = 0$ then the following are equivalent

(i) $A \oplus B$ is QF-3'',

(ii) A is QF-3''.

Using the method of L. Bican [2], Theorem 11 we obtain the following theorem.

Theorem 23. Let M be a finitely embedded left R-module, $S = Hom_R(M,M)$ and $N \in$ R-mod. If M_S is flat and N is QF-3'' then $Hom_R(M,N)$ is a QF-3'' left S-module.

Proof. Let us denote $X = Hom_R(M,N)$. If M_S is flat then ${}_SM^* = Hom_R(M, F(N))$ is injective and consequently $E({}_SX) \subseteq {}_SM^*$. Now if ${}_SY$ is a finitely generated submodule of $E({}_SX)$ then ${}_SY = \sum_{i=1}^{n} Sf_i$, for some $f_i \in {}_SM^*$, $i \in \{1,2,\dots,n\}$. Further

$\operatorname{Im} f_i$ is finitely embedded, $i \in \{1,2,\dots,n\}$ since ${}_R M$ is finitely embedded and hence ${}_S Y \cong \operatorname{Hom}_R(M,Z)$, where Z is a finitely generated submodule of $E(N)$. As it is easy to see $p^{\{{}_S X\}}({}_S Y) \cong p^{\{{}_S X\}}(\operatorname{Hom}_R(M,Z)) = 0$ since $p^{\{N\}}(Z) = 0$ and consequently ${}_S X$ is QF-$3''$.

Corollary 24. Let R, S be Morita equivalent rings via $F = \operatorname{Hom}_R(P,-)$. If ${}_R Q$ is QF-$3''$ then ${}_S F(Q)$ is QF-$3''$.

Corollary 25. Let R and S be Morita equivalent rings via $F = \operatorname{Hom}_R(P,-)$. Then F induces one-to-one correspondence between the isomorphism classes of QF-$3''$ R-modules and QF-$3''$ S-modules.

Corollary 26. If R and S are Morita equivalent rings then R is left QF-$3''$ if and only if S is left QF-$3''$.

Proposition 27. Let $Q \in R$-mod. If every cyclic submodule of Q is QF-$3''$ then Q is QF-$3''$.

Proof. It follows immediately from Proposition 9.

Corollary 28. The following are equivalent

(i) every left R-module is QF-$3''$,

(ii) every cyclic left R-module is QF-$3''$.

Proposition 29. Let R be either left or right semiartinian. Then the following assertions are equivalent

(i) every left R-module is QF-$3''$,

(ii) R is a left V-ring,

(iii) every right R-module is QF-$3''$,

(iv) R is a right V-ring.

Proof. (i) implies (ii) and (iii) implies (iv). It follows from Proposition 12. For the rest see [15].

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