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The volume conjecture and four-dimensional hypersurfaces

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Abstract: In this note we prove the volume conjecture by A. Gray and L. Vanhecke for the four-dimensional hypersurfaces of $\mathbb{R}^5$ with the exception of a subclass of hypersurfaces satisfying a non-trivial geometric inequality.

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Let us consider the following "volume condition":

(V): For an analytic Riemannian manifold $(M, g)$, suppose that any geodesic ball in $(M, g)$ of sufficiently small radius $r > 0$ has the same volume as the Euclidean ball of the same dimension and radius.

The volume conjecture by A. Gray and L. Vanhecke, [2], then says that $(M, g)$ should be locally Euclidean. The volume conjecture has been proved in many important situations, for example, for all manifolds of dimension $n \leq 3$, for manifolds with non-positive, or non-negative Ricci curvature, for the products of surfaces, for the products of classical symmetric spaces, and so on. Little is known about the 4-dimensional Riemannian manifolds with the exception of the case when the metric is Ricci-parallel.
In all these results, what has been really used is not the strong condition (V) but only the information contained in the second order - and the fourth order term of the power-series expansion for the volume of a geodesic ball (with respect to its radius \( r \)). In other words, the following weaker condition has been used as the start point:

\((V'): \) The volume of any small geodesic ball in \((M, g)\) coincides with the volume of the corresponding Euclidean ball up to a remainder term of the form \( O(r^5) \).

The purpose of this Note is to prove the following:

**Theorem.** Let \( M_4 \) be a four-dimensional analytic hypersurface of \( \mathbb{E}^5 \) satisfying the weak volume condition \((V')\).

Then either \( M_4 \) is locally Euclidean, or we have the inequality

\[
(1) \quad -6,9456... \leq (K/h^4) \leq -3,9288...
\]

where \( K \) or \( h \) denotes the Gauss-Kronecker curvature, or the mean curvature of \( M_4 \), respectively.

**Proof.** We shall start with some preparations. For any Riemannian manifold \((M, g)\), let us denote by \( R, \varphi, \tau \) the Riemannian curvature tensor, the Ricci tensor and the scalar curvature of \((M, g)\), respectively. According to (1),(2), the condition \((V')\) is equivalent to the following couple of conditions:

\[
\begin{align*}
(2) & \quad \tau = 0, \\
& \quad 3\|R\|^2 = 8 \|\varphi\|^2,
\end{align*}
\]

where \( \|R\| \) and \( \|\varphi\| \) denotes the norm of \( R \) and \( \varphi \), respectively.
Consider a hypersurface $M \subset \mathbb{R}^{n+1}$ ($n \geq 4$) equipped with the induced Riemannian metric. At any fixed point $p \in M$, let $\lambda_1, \ldots, \lambda_n$ denote the eigenvalues of the second fundamental form, and $s_1, s_2, \ldots, s_n$ the corresponding elementary symmetric functions.

**Lemma.** At any point $p \in M$, the conditions a),b) from (2) are equivalent to the following conditions for the elementary symmetric functions:

a') $s_2 = 0$,

b') $s_1 s_3 = 7s_4$.

**Proof of the Lemma.** Let $p_k$, $k=1,2,\ldots$, denote the sum of the $k$-th powers of the eigenvalues $\lambda_1$. We shall use the following formulas by Newton (cf. [4]):

\[ P_1 = s_1 \]
\[ P_2 = s_1 P_1 - 2s_2 \]
\[ P_3 = s_1 P_2 - s_2 P_1 + 3s_3 \]
\[ P_4 = s_1 P_3 - s_2 P_2 + s_3 P_1 - 4s_4. \]

Hence we obtain immediately

\[ P_2 = (s_1)^2 - 2s_2, \]
\[ P_3 = (s_1)^3 - 3s_1 s_2 + 3s_3, \]
\[ P_4 = (s_1)^4 - 4(s_1)^2 s_2 + 4s_1 s_3 + 2(s_2)^2 - 4s_4. \]

Let us choose an orthonormal basis $\{e_1, \ldots, e_n\}$ of the tangent space $T_p M$ which diagonalizes the second fundamental form (the "shape operator") $S$; then $S_{ij} = \delta_{ij} \lambda_i$ for $i,j = 1, \ldots, n$. We have the Gauss equations

\[ R_{ijkl} = S_{ik} S_{jl} - S_{ik} S_{jk}, \quad i,j,k,l = 1, \ldots, n. \]
Hence \( R_{ij} = R_{ji} = -R_{ji} = -R_{ij} = \lambda_i \lambda_j \) for any \( i \neq j \), and \( R_{ijkl} = 0 \) whenever at least 3 indices are different.

Further, \( \xi_{ii} = \sum_{j=1}^{n} R_{ij} = \frac{\sum_{j=1}^{n} \lambda_i \lambda_j}{i} = 0 \), \( i = 1, \ldots, n \)
and \( \xi_{ij} = 0 \) for all \( i, j = 1, \ldots, n, i \neq j \). Finally,
\[ \gamma = \sum_{i=1}^{n} \xi_{ii} = (s_1)^2 - p_2. \]

From the Newton's formulae we see that \( \gamma = 0 \) is equivalent to \( s_2 = 0 \). Now, we have
\[ \|R\|^2 = \sum_{i=1}^{n} (R_{ij}^2) = \sum_{i=1}^{n} (\lambda_i \lambda_j) = 2(p_2^2 - p_4), \]
i.e.,
(3) \[ \|R\|^2 = 8s_4 + 4s_2^2 - 8s_1s_3, \]

and
\[ \|\varphi\|^2 = \sum_{i=1}^{n} (\varphi_{ii})^2 = \frac{\sum_{i=1}^{n} \lambda_i^2}{4} \left( s_1^2 - 2s_1s_3 + s_4 \right) = s_1p_2 - 2s_1p_3 + p_4 = -2s_1s_3 + 2s_2 - 4s_4. \]
The relation \( 8\|\varphi\|^2 = 3\|R\|^2 \) then yields \( s_4 = \frac{s_1s_3}{7} + \frac{s_2^2}{14} \).

Hence the result follows.

**Proof of the Theorem.** Let us recall the definition of the Gauss-Kronecker curvature and the mean curvature for a hypersurface \( M_4 \subset E^5 \) (cf. [3]). Here we have \( K = s_4 = \lambda_1 \lambda_2 \lambda_3 \lambda_4, \ h = s_1/4 \). From (3) we get \( \|R\|^2 = -48K \), and because \( K = \frac{4}{3}h s_3 \), we see that any of the relations \( K = 0, \ h = 0 \) implies that \( M_4 \) is locally Euclidean.

Suppose now that \( M_4 \) is not locally Euclidean and con-
Consider the characteristic equation of the second fundamental form:

\[ x^4 - a_1x^3 + s_2x^2 - s_3x - s_4 = 0. \]

We shall recall in brief the theory of a biquadratic equation. Consider the equation

\[ x^4 + a_1x^3 + a_2x^2 + a_3x + a_4 = 0 \]

and put

\[
\begin{align*}
    p &= a_2 - \frac{1}{8}a_1^2 \\
    q &= a_3 - \frac{1}{2}a_1a_2 + \frac{1}{8}a_1^3 \\
    r &= a_4 - \frac{1}{4}a_1a_3 + \frac{1}{16}a_1^3a_2 - \frac{3}{256}a_1^4
\end{align*}
\]

The so-called cubic resolvent of the equation (4) is given by

\[
t^3 + 2pt^2 + (p^2 - 4r)t - q^2 = 0.
\]

The discriminant \( D \) of the equation (4) can be written in the form

\[
D = 16p^4r - 4p^3q^2 - 128p^2r^2 + 144prq^2 + 256r^3 - 27q^4.
\]

We have \( D = \prod_{1 \leq i < j \leq 4} (\lambda_i - \lambda_j)^2 \).

Now, the general theory (see [6]) says that the equation (4) has 4 simple real roots if and only if

\[ D > 0, \; p < 0, \; p^2 - 4r > 0. \]

The equality \( D = 0 \) corresponds to the case of a multiple root.

In our case we have

\[
\begin{align*}
    p &= -\frac{3}{8}s_1^2, \\
    q &= -s_3 - \frac{1}{8}s_1^3, \\
    r &= -\frac{3}{4}s_4 - \frac{3}{256}s_1^4.
\end{align*}
\]
Hence $p < 0$ iff $h - j = 0$, and $p^2 - 4r > 0$ iff $\frac{1}{16}s_1^4 + s_4 > 0$, i.e., $-16 < (K/h^4)$.

After a long but routine calculation we get

$$D = -\frac{243s_4^2s_1^4}{16s_3^4} + \frac{81s_4s_3^5}{16s_3^4} - \frac{27s_4^2s_1^6}{16s_3^4} - 108s_3^4 + \frac{81s_4s_3^2s_1^2}{16s_3^4} - \frac{27s_3^3s_1^3}{16s_3^4} - 27s_3^4.$$  

Substituting now $s_1 = 4h$, $s_3 = \frac{7K}{4h}$, $s_4 = K$, we get

$$D = -27K^2h^4 \left[\left(\frac{7}{16}\right)^4(K/h^4)^2 + \left(\frac{102}{16}\right)^2(K/h^4) + 1\right]$$

and hence the condition $D \geq 0$ implies

$$\left(\frac{7}{16}\right)^4(K/h^4)^2 + \left(\frac{102}{16}\right)^2(K/h^4) + 1 \leq 0.$$  

This is the case if and only if

$$-(51 + \sqrt{200})(16/49)^2 \leq K/h^4 \leq -(51 - \sqrt{200})(16/49)^2$$

which is the wanted inequality (1). The relation $-16 < K/h^4$ is a consequence of the above, thus it cannot bring in any new restrictions for our invariants. It can be also checked that in the case $D = 0$ our equation (4) has only real roots, too.

**Remark.** The inequality (1) in our theorem has an intrinsic meaning. In fact, because $K \neq 0$, the second fundamental form is non-degenerate and thus, following [5], it is uniquely determined by the metric of $M_4$ (upto a sign). Thus $K/h^4$ is a Riemannian invariant of $M_4$. It remains an open problem whether the inequality (1) is compatible with the strong volume condition (V), or not.
References


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