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FIXED POINT THEOREMS FOR MULTIVALUED MAPPINGS
LE VAN HOT

Abstract: We prove new fixed point theorems for multivalued mappings. Moreover, we construct a simple example which shows that the conjecture of J. P. Penot, stated in [8], is false.

Key words: Metric space, Banach space, fixed point theorems, multivalued mappings.

Classification: Primary 47H10, 47H15
Secondary 54C60

1. A fixed point theorem for multivalued mappings in complete metric spaces.

Let $M$ be a metric space with metric $d$, $A, B$ being subsets of $M$, $x_0 \in M$. Put: $d(x_0, A) = \inf \{d(x_0, x) : x \in A\}$, $D(A, B) = \{\lambda > 0 : A \subseteq V_\lambda (B) \text{ and } B \subseteq V_\lambda (A)\}$, $\sup \{d(y, A) : y \in B\}$, where $V_\lambda (A) = \{y \in M, d(y, Y) \leq \lambda \}$ for $\lambda > 0$.

Definition 1. Let $M$ be a metric space with metric $d$. We say that a map $f : M \to M$ satisfies Caristi's condition if there exists a lower semicontinuous function $h : M \to \mathbb{R}_+ = [0, \infty)$ such that $d(x, f(x)) \leq h(x) - h(f(x))$ for all $x \in M$.

Theorem 1. Let $M$ be a complete metric space, $F : M \to M$ be a multivalued mapping of $M$ into the family of all nonempty compact subsets of $M$ such that $D(F(x), F(y)) < d(x, y)$ for all
Suppose that there exists a single-valued map \( f : M \rightarrow M \) satisfying the Caristi's condition such that:

1) \( d(x, f(x)) \leq \inf \{ d(f^n(x), f(f^n(x)) : n = 1, 2, \ldots \} \)

for all \( x \in M \), where \( f^n(x) = (f \circ f \circ \ldots \circ f)(x) \), \( n \)-times

2) \( K = \{ x \in M, f(x) = x \} \) is precompact.

Then \( F \) has a fixed point in \( M \).

**Proof.** We claim that for each \( z \in M \) there exists a \( z_0 \in K \) such that \( d(z_0, F(z_0)) \leq d(z, F(z)) \). Let \( h : M \rightarrow \mathbb{R}_+ \) be a lower semicontinuous function such that \( d(x, f(x)) \leq h(x) - h(f(x)) \) for all \( x \in M \). We write \( x \preceq y \) iff \( d(x, y) \leq h(x) - h(y) \). Then \( \preceq \) is a partial order on \( M \). Let \( z \) be an arbitrary fixed point in \( M \). Put \( M_z = \{ x \in M : d(x, f(x)) \leq d(z, F(z)) \} \). Then \( M_z \) is a non-empty \((z \in M_z)\) closed subset of \( M \), since \( d(x, f(x)) \) is a continuous function on \( M \). Therefore \( M_z \) is complete. Using the same argument as in [8] one can prove that there exists a maximal element \( z_0 \) in \( M_z \) (i.e. if \( x \in M_z \) and \( x \succeq z_0 \) then \( x = z_0 \)).

Suppose that there exists an \( n \in \mathbb{N} \) such that

\[ d(f^n(z_0), F(f^n(z_0))) \leq d(z_0, F(z_0)) \leq d(z, F(z)) \]

Then \( f^n(z_0) \in M_z \). On the other hand, we have:

\[ d(z_0, f(z_0)) \leq h(z_0) - h(f(z_0)), \quad d(f(z_0), f^2(z_0)) \leq h(f(z_0)) - h(f^2(z_0)), \ldots, \quad d(f^{n-1}(z_0), f^n(z_0)) \leq h(f^{n-1}(z_0)) - h(f^n(z_0)). \]

Hence

\[ d(z_0, f^n(z_0)) \leq \sum_{i=1}^{n} d(f^{i-1}(z_0), f^i(z_0)) \leq h(z_0) - h(f^n(z_0)), \]

where \( f^0(z_0) = z_0 \). This implies \( f^n(z_0) \succeq z_0, f^n(z_0) \in M_z \). Hence \( f^n(z_0) = z_0 \) and it is clear that \( f(z_0) = z_0 \in K \cup M_z \).

Now suppose that \( d(f^n(z_0), F(f^n(z_0))) > d(z_0, F(z_0)) \) for all \( n \). Then there exists a subsequence \( \{ n_i \} \) such that

\[ \lim d(f^{n_i}(z_0), F(f^{n_i}(z_0))) = d(z_0, F(z_0)). \]

It is easy to see
that \( f^n(z_0) \) is a Cauchy sequence in \( M \). Then there exists a point \( z_\infty \in M \) such that \( z_\infty = \lim f^n(z_0) \), since \( M \) is complete. Hence
\[
d(z_0, z_\infty) = \lim d(z_0, f^n(z_0)) = h(z_0) - \lim h(f^n(z_0)) = h(z_0) - h(z_\infty),
\]
\[
d(z_\infty, F(z_\infty)) = \lim d(f^n(z_0), F(f^n(z_0))) = d(z_0, F(z_0)) = d(z, F(z)).
\]
This means that \( z_\infty \in M \) and \( z_\infty \geq z_0 \). Therefore \( z_\infty = z_0 \) and \( h(z_\infty) = h(f(z_0)) = h(z_0) \). Hence \( d(f(z_0), F(f(z_0))) = d(z_0, F(z_0)) \).

This contradicts the assumption
\[
d(f^n(z_0), F(f^n(z_0))) > d(z_0, F(z_0)) \text{ for all } n=1,2,\ldots.
\]
This proves our claim.

It is easy to see that \( \inf \{ d(x, F(x)) : x \in M \} = \inf \{ d(x, F(x)) : x \in \overline{K} \} \). Since \( \overline{K} \) is compact, there exists a point \( x_0 \in \overline{K} \) such that \( d(x_0, F(x_0)) = \inf \{ d(x, F(x)) : x \in M \} \). If \( r = d(x_0, F(x_0)) > 0 \), take a \( y \in F(x_0) \) such that \( d(x_0, y) = d(x_0, F(x_0)) = r \). Then \( d(y, F(y)) \leq d(F(x_0), F(y)) < d(x_0, y) = r \).

This contradicts the assumption
\[
d(x_0, F(x_0)) = \inf \{ d(x, F(x)) : x \in M \}.
\]
Hence \( d(x_0, F(x_0)) = 0 \) and \( x_0 \in F(x_0) \). This completes the proof.

**Remark:** In [8] J.P. Penot has stated the following problem: Let \( M \) be a complete metric space, \( h: M \to R_+ \) be a lower semicontinuous function and \( F: M \to M \) be a multivalued mapping of \( M \) into the family of all nonempty closed subsets of \( M \) satisfying the following condition:
\[
d(x, F(x)) \leq h(x) - \inf \{ h(y) : y \in F(x) \} \geq 0.
\]
Does \( F \) have a fixed point in \( M \)?

The following simple example shows that this conjecture
Proposition 1. Let $M$ be a complete metric space, $h: M \to \mathbb{R}_+$ be a lower semicontinuous function, $F: M \to M$ be a multivalued mapping which maps $M$ into the family of all nonempty closed subsets of $M$. Suppose $F$ satisfies the following condition

$$\inf \{ d(x,y) + h(y) : y \in F(x) \} = d(x, F(x)) = h(x)$$

for all $x \in M$. Then $F$ has a fixed point in $M$.

Proof. We claim that for each $x \in M$ there exists an $f(x) \in F(x)$ such that $d(x, f(x)) \leq 2h(x) - 2h(f(x))$. If $d(x, F(x)) = 0$, put $f(x) = x$. If $d(x, F(x)) > 0$, then

$$d(x, F(x)) + \inf \{ d(x,y) + 2h(y) : y \in F(x) \} \leq 2 \inf \{ d(x,y) + h(y) : y \in F(x) \} = 2h(x).$$

It follows that $\inf \{ d(x,y) + 2h(y) : y \in F(x) \} < 2h(x)$. Then there exists a point $f(x) \in F(x)$ such that $d(x, f(x)) + 2h(f(x)) \leq \leq 2h(x)$. This proves our claim.

According to Ćirić's Theorem there exists a point $x_0 \in M$ such that $x_0 = f(x_0) \in F(x_0)$. This completes the proof.

Corollary 1 (S. B. Nadler [7]). Let $M$ be a complete metric space. If $F: M \to M$ is a multivalued contraction mapping which maps $M$ into the family of all nonempty closed subsets of $M$, then $F$ has a fixed point.

Proof. Let $D(F(x), F(y)) = kd(x, y)$, where $0 \leq k < 1$. Put
\[ h(x) = \frac{1}{1-k} d(x, \mathcal{F}(x)) \]. Then
\[
\inf \sum d(x, y) + h(y) : y \in \mathcal{F}(x) = \inf \sum d(x, y) + \frac{1}{1-k} d(y, \mathcal{F}(y)) : y \in \mathcal{F}(x)
\]
\[
\leq \inf \sum d(x, y) + \frac{1}{1-k} k d(x, y) : y \in \mathcal{F}(x)
\]
\[
= \inf \sum d(x, y) + \frac{1}{1-k} d(x, \mathcal{F}(x)) = h(x).
\]

By Proposition 1, \( \mathcal{F} \) has a fixed point in \( M \).

**Corollary 2.** Let \( M, h, \mathcal{F} \) be as in Proposition 1.

1. If \( d(x, \mathcal{F}(x)) \leq h(x) - \sup h(y) : y \in \mathcal{F}(x) \), then \( \mathcal{F} \) has a fixed point in \( M \).

2. If \( D(-x, \mathcal{F}(x)) \geq h(x) - \inf h(y) : y \in \mathcal{F}(x) \), then there exists an \( x_0 \in M \) such that \( f(x_0) = -x_0 \).

**Proof.** It is clear that \( \mathcal{F} \) has a fixed point in \( M \), because \( \inf \sum d(x, y) + h(y) : y \in \mathcal{F}(x) \geq d(x, \mathcal{F}(x)) + \sup h(\mathcal{F}(x)) \) and
\[
\inf \sum d(x, y) + h(y) : y \in \mathcal{F}(x) \geq D(-x, \mathcal{F}(x)) + \inf h(y) : y \in \mathcal{F}(x).
\]

To prove 2, it is sufficient to note that for each \( x \in M \) there exists a point \( f(x) \in \mathcal{F}(x) \) such that
\[
D(-x, \mathcal{F}(x)) \leq h(x) - \inf h(y) : y \in \mathcal{F}(x) \leq 2h(x) - 2h(f(x)).
\]

By Caristi's Theorem there exists a point \( x_0 \in M \) such that \( x_0 = f(x_0) \). Then \( D(-x_0, \mathcal{F}(x_0)) \leq 2h(x_0) - 2h(f(x_0)) = 0 \). It follows that \( \mathcal{F}(x_0) = -x_0 \). This completes the proof.

7. **A fixed point theorem for multivalued mappings in Banach spaces**

**Definition 2.** Let \( X, Y \) be topological spaces, \( \mathcal{F} : X \to 2^Y \) be a multivalued mapping. We say that \( \mathcal{F} \) is upper semicontinuous at \( x \in X \) if for each open set \( G \subset Y \), \( \mathcal{F}(x) \cap G \) there exists a neighborhood \( U \) of \( x \) such that for each \( x' \in U \) we have \( \mathcal{F}(x') \subseteq G \).
Theorem 2. Let $X$ be a Banach space, $C \subseteq X$ be a convex closed nonempty bounded subset of $X$, $f: C \rightarrow C$ be a multivalued nonexpansive mapping which maps into the family of all nonempty convex closed subsets of $C$. Suppose that there exist a function $\omega: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ which is nondecreasing and $\omega(t) > 0$ for all $t > 0$, a function $\varphi: C \rightarrow \mathbb{R}$ weakly continuous at $0$, $\varphi(0) > 0$ and a mapping $\psi: C \rightarrow \mathcal{C}(X^*)$, where $\mathcal{C}(X^*)$ denotes the family of all nonempty closed subsets of the dual space $X^*$, weakly-strongly upper-semicontinuous at $0$, $\psi(0)$ is compact, such that

$$d(x, f(x)) + d(y, f(y)) \geq \omega(\|x - y\|) \varphi(x - y) - \psi_s(x - y)$$

for all $x, y \in C$, where $\psi_s(x) = \sup \{\langle x^*, x \rangle | x^* \in \psi(x)\}$. Then $f$ has a fixed point in $C$.

Proof. By the boundness of $C$, there exists a number $M > 0$ such that $C \subseteq B_M = \{x \in X : \|x\| \leq M\}$. Hence $C \subseteq B_{2M}$. By the standard argument there exists a sequence $\{x_n\} \subseteq C$ such that $d(x_n, f(x_n)) < \frac{1}{n}$ for each $n \in \mathbb{N}$. Since $\{x_n\}$ is bounded in $X$, $\{x_n\}$ is weakly precompact. Then there exists a weakly Cauchy subnet $\{x_{\sigma(i)}\}_{i \in I}$ of $\{x_n\}$ where $\sigma: I \rightarrow \mathbb{N}$. Then it is clear that the net $\{u_{i,j}\}_{(i,j) \in I \times I}$ where $u_{i,j} = x_{\sigma(i)} - x_{\sigma(j)}$ converges weakly to $0$.

We claim that $\lim \|u_{i,j}\| = 0$. Suppose that it is false. There exists a number $k > 0$ such that for any $(i,j) \in I \times I$ there exists an $(i', j') \in I \times I$, $(i', j') \neq (i,j)$ and $\|u_{i,j}\| \geq k$. Since $\varphi$ is weakly continuous at $0$ we have $\lim \varphi(u_{i,j}) = \varphi(0) = k > 0$. Let $\tau: X \rightarrow X^{**}$ be a canonical embedding map of $X$ into its bidual space $X^{**}$. Since $\{\tau(u_{i,j})\}$ is bounded in $X^{**}$, $\{\tau(u_{i,j})\}$ is an equicontinuous family of mappings.
from \((X^*, \| \cdot \|)\) into \(\mathbb{R}\). Since \(\{\varphi(u_{1,j})\}\) converges pointwise to \(\varphi(0)\) on \(X^*\) and \(\varphi(0)\) is a compact subset of \(X^*\) by Theorem 4.5 [9, chapt. III] it follows that \(\{\varphi(u_{1,j})\}\) converges uniformly to \(\varphi(0)\) on \(\varphi(0)\). Then there exists an index \((i_0, j_0) \in I \times I\) such that for \((i, j) \in I \times I\), \((i, j) \geq (i_0, j_0)\) we get

\[\varphi(u_{i,j}) \geq \frac{3}{4} \cdot k\] and \(\|x^* u_{i,j}\| = \|x^* u_{1,j}\| \leq \frac{1}{8} k \alpha(\varphi(r))\) for all \(x^* \in \varphi(0)\). Since \(\varphi\) is weakly-strongly upper-semicontinuous at \(\varphi(0)\) and \(\{u_{i,j}\}\) converges weakly to \(\varphi(0)\), there exists an index \((i_1, j_1) \in I \times I\), \((i_1, j_1) \geq (i_0, j_0)\) such that:

\[
\varphi(u_{i_1,j_1}) \leq \varphi(\varphi(0)) + \frac{k \alpha(\varphi(r))}{16M} B_1(\varphi(0)), \quad \text{where} \quad B_1 = \{x^* \in X^* : \|x^*\| \leq 1\}
\]

for all \((i, j) \in I \times I, (i, j) \geq (i_1, j_1)\). Then

\[
\varphi_s(u_{i,j}) = \sup \{\|x^* u_{i,j}\| : x^* \in \varphi(0) + \frac{k \alpha(\varphi(r))}{16M} B_1(\varphi(0))\} \leq \sup \{\|x^* u_{i,j}\| : x^* \in \varphi(0) + \frac{k \alpha(\varphi(r))}{16M} B_1(\varphi(0))\} + \frac{k \alpha(\varphi(r))}{4}
\]

for all \((i, j) \in I \times I, (i, j) \geq (i_1, j_1)\).

Take \(n, m \in \mathbb{N}\) such that \(\frac{1}{n} + \frac{1}{m} \leq \frac{k \alpha(\varphi(r))}{2}\). Choose \(i_2 \in I, i_2 \geq i_1, i_2 \geq j_1\) such that \(\varphi(i) \geq \max \{n, m\}\) for all \(i \in I, i \geq i_2\). Take \((i_3, j_3) \in I \times I, (i_3, j_3) \geq (i_2, j_2)\) such that \(\|u_{i_3,j_3}\| \geq r\). Then

\[
d(\varphi(0)(i_3), \varphi(0)(j_3)) + d(\varphi(0)(j_3), \varphi(0)(j_3)) = \varphi_s(u_{i_3,j_3}) - \varphi(0)(i_3) - \varphi(0)(j_3).
\]

Hence

\[
\frac{1}{n} + \frac{1}{m} \leq \frac{1}{\delta(0)(j_3)} + \frac{1}{\delta(0)(j_3)} \leq d(\varphi(0)(i_3), \varphi(0)(j_3)) + d(\varphi(0)(j_3), \varphi(0)(j_3)) - \frac{2}{3} k \alpha(\varphi(r)) - \frac{k \alpha(\varphi(r))}{4} = \frac{1}{2} k \alpha(\varphi(r)).
\]

This contradicts \(\frac{1}{n} + \frac{1}{m} \leq \frac{1}{2} k \alpha(\varphi(r))\) and this proves our claim.
Since \( \lim \| u_{i,j} \| = 0 \), it follows that \( \{ x_{\varphi(1)}^j \} \) is a Cauchy net in the strong topology. Therefore \( \{ x_{\varphi(1)}^j \} \) converges strongly to an \( x \in C \). Then for \( i \leq I \), we have

\[
d(x, F(x)) \leq \| x - x_{\varphi(1)} \| \cdot d(x_{\varphi(1)}, F(x)) + \gamma \| F(x) - F(x_{\varphi(1)}) \| \cdot d(x, F(x)) \leq 2 \| x - x_{\varphi(1)} \| + \frac{1}{\varphi(1)}.
\]

Hence \( d(x, F(x)) = 0 \). It follows that \( x \in F(x) \) and this completes the proof.

References


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