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ALMOST LOWER SEMICONTINUOUS MULTIFUNCTIONS AND THE SOUSLIN-GRAPH THEOREM

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Abstract: Almost continuous mappings and almost lower semicontinuous multifunctions are investigated. A Souslin-graph theorem for multihomomorphisms with values in an analytic space is proved.

Key words: Multifunction, almost lower semicontinuity, Souslin-graph.

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1. Introduction. The term "almost continuity" is used here in the sense of Bradford and Goffman [4]. We show that each almost continuous mapping having the Baire property and taking values in a regular space is continuous (Theorem 4). It follows that each almost continuous mapping having a Souslin graph and taking values in an analytic space is continuous (Theorem 6).

We define and investigate "almost lower semicontinuity" of multifunctions. Under category type assumptions certain multifunctions possess automatically this property (Theorems 1,2,3).

Let $F:G \longrightarrow H$ be a multihomomorphism with $F^{-1}(H) = G$. If G is of second category and H is separable or Lindelöf, then F is lower semicontinuous iff it is lower-Baire (Theorem 5). If G is inductively generated by second category groups and

H is analytic, then F is lower semicontinuous provided it has a Souslin graph (Theorem 7). This version of the Souslin-graph theorem is based on ideas due to Frolík [7] and [6].

2. Almost lower semicontinuous multifunctions. Almost continuous mappings were considered first, as it seems, by Blumberg [3] and Block and Cargal [2], under unlike names. The term "almost continuity" was used by Bradford and Goffman [4].

Let X and Y be topological spaces and f a mapping of X to Y ($f: X \longrightarrow Y$). Given $x \in X$, f is said to be almost continuous at x if for each open set V in Y containing $f(x)$, $x \in \operatorname{Int} D(f^{-1}(V))$. Here $D(E)$, where $E \subset X$, denotes (as in [10]) the set of all points x' of X that are of second category in X relative to E (i.e. $U \cap E$ is of second category in X for each open $U \ni x'$). This definition of almost continuity is equivalent to those given in the above-mentioned papers, and can be extended, in a natural way, to multifunctions. By a multifunction F of X to Y ($F: X \longrightarrow Y$) we mean a function which to every point $x \in X$ assigns a subset $F(x)$ of Y (not necessarily closed or nonempty).

Definition. A multifunction $F: X \longrightarrow Y$ is almost lower semicontinuous at x if for every open set V in Y

$$x \in F^{-1}(V) \text{ implies } x \in \operatorname{Int} D(F^{-1}(V)).$$

Here the inverse image $F^{-1}(V)$ denotes, as always, the set of all x' satisfying $F(x') \cap V \neq \emptyset$.
The set of all points x of X such that F is almost lower semicontinuous at x will be denoted by $L_{a}(F)$; in case $L_{a}(F) = X$, F will be called almost lower semicontinuous. Thus, F is almost lower semicontinuous if and only if for every open set V

in Y

$$F^{-1}(V) \subset \operatorname{Int} D(F^{-1}(V)).$$

Let $L(F)$ stand for the set of all points $x \in X$ such that F is lower semicontinuous at x, i.e. $x \in \operatorname{Int} F^{-1}(V)$ for all open $V \subset Y$ intersecting $F(x)$. Notice that

$$L_a(F) \cap F^{-1}(Y) \subset \operatorname{Int} D(F^{-1}(Y)) \subset \operatorname{Int} D(X)$$

and

$$L(F) \cap F^{-1}(Y) \cap \operatorname{Int} D(X) \subset L_a(F),$$

while obviously $X \setminus F^{-1}(Y) \subset L(F) \cap L_a(F)$. In particular, if X is a Baire space (i.e. $X = D(X)$), then $L(F) \subset L_a(F)$. If F is almost lower semicontinuous, then $F^{-1}(Y)$ is a Baire space (in itself).

The usefulness of the property of almost lower semicontinuity stems from the fact that it is automatically satisfied under some category-type assumptions, while, on the other hand, it is a convenient starting point to the Souslin-graph, closed graph, open mapping and Blumberg theorems.

The following theorem extends some observations from [3], [2] and [4].

<u>Theorem 1</u>. Let F be a multifunction of X to Y. If the space Y is second-countable, then

(i) The set $L_a(F)$ is residual in X;

(ii) the restriction $F|L_a(F): L_a(F) \longrightarrow Y$ is almost lower semicontinuous. More generally, for each residual set $A \subset L_a(F)$, $F|A$ is almost lower semicontinuous.

<u>Proof</u>. (i) Let $\{V_n\}$ be a base for Y. A point $x \in X$ is not in $L_a(F)$ if and only if there is n such that $x \in F^{-1}(V_n)$

and $x \notin \operatorname{Int} D(F^{-1}(V_n))$. Thus

$$L_a(F) = X \setminus \bigcup_{n=1}^{\infty} [F^{-1}(V_n) \setminus \operatorname{Int} D(F^{-1}(V_n))].$$

Each set of the form $E \setminus \operatorname{Int} D(E)$ is of first category because $D(E) \setminus \operatorname{Int} D(E)$ is closed co-dense and $E \setminus D(E)$ is of first category by the Banach category theorem (cf. [10]). Hence $L_a(F)$ is residual.

(ii) Let $A \subset L_a(F)$ be residual in X. Put $E = F^{-1}(V_n)$. Then

$$A \cap D(E) = A \cap D(A \cap E) \subset D_A(A \cap E)$$

and

$$x \in A \cap \operatorname{Int} D(E) \subset \operatorname{Int}_A(A \cap D(E)) \subset \operatorname{Int}_A D_A(A \cap E) \text{ for } x \in A \cap E,$$

which shows almost lower semicontinuity of $F|A$.

By a graph of a multifunction $F: X \longrightarrow Y$ we mean the set

$$Gr\ F = \{(x,y) : y \in F(x)\} \subset X \times Y.$$

In the following the letters G, H stand for topological groups. We say that $F: G \dashrightarrow H$ is a multihomomorphism if $Gr\ F$ is a subgroup of $G \times H$. For multihomomorphisms we have the following simple criterion of almost lower semicontinuity.

<u>Lemma 1</u>. A multihomomorphism $F: G \longrightarrow H$ is almost lower semicontinuous if (and only if) for each neighbourhood V of e_H the inverse image $F^{-1}(V)$ is of second category in G.

<u>Proof</u>. Let V be a symmetric neighbourhood of e_H and put $E = F^{-1}(V)$ and $U = \operatorname{Int} D(E)$. Since $E \setminus U$ is a first category set in G (by the Banach category theorem; see the previous proof) and E is of second category (by the hypothesis), the set U is non-empty. This implies that $e_G \in \operatorname{Int} D(F^{-1}(V^2))$ because

$$e_G \in U^2 \subset \operatorname{Int}\ ([D(F^{-1}(V))]^2) \subset \operatorname{Int}\ D([F^{-1}(V)]^2) = \operatorname{Int}\ D[F^{-1}(V^2)].$$

Thus $e_G \in L_a(F)$. If now $x \in F^{-1}(V)$, where V is open in H, then $e_G \in F^{-1}(Vy^{-1})$, where $y \in F(x) \cap V$; hence $e_G \in \operatorname{Int}\ D[F^{-1}(Vy^{-1})]$, and so $x \in \operatorname{Int}\ D([F^{-1}(Vy^{-1})]x) = \operatorname{Int}\ D[F^{-1}(V)]$; $x \in L_a(F)$.

Now we need a generalization of a lemma of Pettis [13] for multihomomorphisms. First a definition ([13]). A subset E of H is σ-bounded in H if for every neighbourhood V of e_H there exists a sequence $\{y_n\} \subset E$ such that $E \subset \bigcup_{n=1}^{\infty} y_n V \cup V y_n$. Each separable or Lindelöf (in particular, σ-compact) subspace E of H is σ-bounded in H. If H is metrizable, the three notions (σ-boundedness, separability and Lindelöf property of $E \subset H$) coincide.

Lemma 2. Let $F: G \to H$ be a multihomomorphism such that $F(G)$ is σ-bounded in H. If $F^{-1}(H)$ is a second category set in G, then so is $F^{-1}(V)$ for any neighbourhood V of e_H.

Proof. Given open $V \ni e_H$, choose $\{y_n\} \subset F(G)$ so that $F(G) \subset \bigcup_{n=1}^{\infty} y_n V \cup V y_n$. Choose $x_n \in F^{-1}(y_n)$. Then $F^{-1}(H) = \bigcup_{n=1}^{\infty} x_n E \cup E x_n$, where $E = F^{-1}(V)$. Hence E is of second category in G.

Lemma 3 ([13]). If H is σ-bounded, then each set $E \subset H$ is σ-bounded in H.

Proof. Let $H = \bigcup_{n=1}^{\infty} y_n V \cup V y_n$, where V is a neighbourhood of e_H and $\{y_n\} \subset H$. Choose $h_n^1 \in E \cap y_n V$ whenever possible ($n \in N_1$) and $h_n^2 \in E \cap V y_n$ whenever possible ($n \in N_2$). Then

$$E \subset \bigcup_{n \in N_1} y_n V \cup \bigcup_{n \in N_2} V y_n \subset \bigcup_{n \in N_1} h_n^1 V^{-1} V \cup \bigcup_{n \in N_2} V V^{-1} h_n^2.$$

Since $V \ni e_H$ was arbitrary, E is σ-bounded.

If now H is σ-bounded, then $F(G)$ is σ-bounded in H

(Lemma 3) and the lemmas Nos. 2 and 1 may be applied, provided $F^{-1}(H)$ is of second category. Thus we get

Theorem 2. Let F be a multihomomorphism of G to H, where H is a σ-bounded group (e.g. separable or Lindelöf). If $F^{-1}(H)$ is of second category in G, then F is almost lower semicontinuous.

For linear multifunctions the assumption of σ-boundedness of the range space may be omitted and the proof reduced. Let S and T be topological vector spaces; $F:S \to T$ is a linear multifunction if Gr F is a linear subspace of $S \times T$.

Theorem 3. Each linear multifunction $F:S \to T$ such that $F^{-1}(T)$ is of second category in S is almost lower semicontinuous.

Proof. Let V be a neighbourhood of 0_T. Since $T = \bigcup_{n=1}^{\infty} nV$, $F^{-1}(T) = \bigcup_{n=1}^{\infty} nF^{-1}(V)$. Hence $F^{-1}(V)$ is of second category in S and we apply Lemma 1.

That is all about "automatic" almost lower semicontinuity. Now we will consider the question, when almost lower semicontinuity (resp. almost continuity) implies lower semicontinuity (resp. continuity). For mappings we have a quite satisfactory answer:

Theorem 4. Let X be a Baire space, and let Y be a regular space (even not necessarily T_0). A mapping $f:X \to Y$ is continuous if (and only if) it is almost continuous and has the Baire property.

Proof. Let $x \in f^{-1}(V)$, where V is open in Y. Choose open set $W \subset Y$ with $f(x) \in W$ and $\overline{W} \subset V$. Since f is almost continuous at x, $x \in U = \operatorname{Int} D(f^{-1}(W))$. Let $u \in U$; we will show that $f(u) \in$ $\in V$. Let Z be an open neighbourhood of $f(u)$. Since f is almost continuous at u, $u \in \operatorname{Int} D(f^{-1}(Z))$. Hence $U \cap f^{-1}(Z)$ is a second category set in X. Since f has the Baire property, there exists an open set $G \subset X$ such that $G \triangle f^{-1}(Z)$ is of first category in X. Now $U \cap G$ is of second category in X. It follows that $f^{-1}(W) \cap G$ is of second category in X. Hence $f^{-1}(W) \cap$ $\cap f^{-1}(Z)$ is of second category in X, which yields $W \cap Z \neq \emptyset$. Thus we have proved that $f(u) \in \overline{W}$.

The theorem cannot be extended to multifunctions, without additional assumptions.

Example 1. Each of the following multifunctions is almost lower semicontinuous and lower-Baire (i.e. $F^{-1}(V)$ has the Baire property whenever $V \subset Y$ is open), but not lower semicontinuous.

(a) $F(x) = \{1\}$ for $x \in I \setminus Q$ and $F(x) = \emptyset$ for $x \in I \cap Q$ ($I = [0,1]$, Q - the rationals); $F: I \longrightarrow I$ is single-valued.

(b) $F(x) = Y$ for $x \in I \setminus Q$ and $F(x) = \{1\}$ for $x \in I \cap Q$, where Y is the discrete space $\{0,1\}$; $F^{-1}(Y) = I$.

(c) (cf. [5]). Let $I = P_1 \cup P_2$, where P_i are dense and co-dense G_δ-sets in I, and let g be the natural mapping of the space $Y = P_1 \oplus P_2$ onto I; g is continuous and almost open (i.e. $g(U) \subset \operatorname{Int} D(g(U))$ for each open $U \subset Y$). Define $F = g^{-1}$.

An analogue of Theorem 4 for multihomomorphisms holds true. To see this, let $F: G \dashrightarrow H$ be an almost lower semicontinuous lower-Baire multihomomorphism, and consider the induced mapping $f: X \dashrightarrow Y$, where $X = F^{-1}(H)$ and $Y = H/F(e_G)$ (Y need not be a T_0-space of a group). The assumptions of Theorem 4 are satisfied (Y is regular by [9; 5.19, 5.20]). Hence f is continuous. Since the quotient mapping $\varphi: H \longrightarrow Y$ is open (cf. [9; 5.17]), this implies lower semicontinuity of $F: X \longrightarrow H$. X is a second category subgroup of G having the Baire property; by the Banach-Kuratowski-Pettis theorem (cf. [1; Theorem 1], [10; 13.XI] and [13; Theorem 1]), X is open in G. Hence $F: G \longrightarrow H$ is lower semicontinuous. Thus, in view of Theorem 2, we get

Theorem 5. Let F be a multihomomorphism of G to H such that $F^{-1}(H)$ is of second category in G. (i) F is lower semicontinuous if (and only if) it is almost lower semicontinuous and lower-Baire. (ii) Suppose the group H is σ-bounded (e.g. separable or Lindelöf). Then F is lower semicontinuous if (and only if) it is lower-Baire.

For linear F, (ii) holds with no assumption on the range vector space (by Theorem 3).

3. Souslin-graph theorem. A T_3-space Y is said to be an analytic space (or a K-Souslin space) if there exists a Polish space X and a compact-valued upper semicontinuous multifunction Φ of X onto Y (Frolík [6]; for some equivalent definitions see [6] and [8]). Each analytic space is a Lindelöf space, hence paracompact and normal (cf. [6] and [8]). By a Souslin set, in a given space, we mean the result of performing the Souslin operation (A) (denoted also S) on a system of closed sets in the space. Since the collection of all sets having the Baire property is closed under the operation

(A), each Souslin set has the Baire property (cf. [10]).

L. Schwartz [15] proved that if S and T are locally convex spaces, S - ultrabornological (i.e. inductive limit of Banach spaces), T - continuous image of a Polish space, then each borel graph linear map $f:S \longrightarrow T$ is continuous and each continuous linear map $g:T \xrightarrow{\text{onto}} S$ is open.

Frolík [7] proved that if G is a vector space which is inductively generated by second category vector spaces and H is an analytic locally convex space, then

(1) each Souslin-graph homomorphism $f:G \longrightarrow H$ is continuous.

Martineau [11] proved, among other results, that if G is a second category analytic group and H is an analytic group, then each continuous homomorphism $g:H \xrightarrow{\text{onto}} G$ is open; Perez Carreras [12] showed that the theorem remains true if G is not necessarily analytic.

In this section we shall show that if G is inductively generated by second category groups and H is an analytic group, then the statements (1) and (2) hold, where

(2) each Souslin-graph homomorphism $g:H \xrightarrow{\text{onto}} G$ is open.

The main tools are Theorem 5 and the following lemma due to Rogers and Willmott [14] (a nice proof is given in Frolík [7; Lemma 1]).

<u>Lemma 4</u>. Let $F:X \longrightarrow Y$ be a multifunction, where Y is an analytic space. If Gr F is a Souslin set in $X \times Y$, then F is upper-Souslin (i.e. $F^{-1}(A)$ is Souslin whenever A is closed), and hence upper-Baire.

Combining the lemma with Theorem 4 we get

Theorem 6. Let X be a Baire space, and let Y be an analytic space. A mapping $f:X \longrightarrow Y$ is continuous if (and only if) it is almost continuous and its graph is a Souslin set in $X \times Y$.

Example 1 (c) shows that an almost lower semicontinuous Souslin-graph multifunction $F:X \longrightarrow Y$ need not be lower semicontinuous, even if X is compact, Y Polish and $\mathrm{Gr}\, F$ closed.

Lemma 5. Each upper Baire multihomomorphism $F:G \longrightarrow H$ is lower-Baire.

Proof. Let φ be the canonical mapping of H onto $H/F(e_G)$. Let V be open in H. Since φ is open and continuous, the set

$$F^{-}(V) = F^{-1}(H) \setminus F^{-1}(\varphi^{-1}[H/F(e_G) \setminus \varphi(V)])$$

has the Baire property in G.

If F is lower-Baire and $F(e_G)$ is compact, then φ is closed (cf. [9; 5.18]) and, consequently, F is upper-Baire. Without the compactness assumption, the converse to Lemma 5 is not true.

Example 2. Let H_0 be a closed normal subgroup of H, $G = H/H_0$, $\varphi:H \rightarrow G$ the canonical quotient mapping and $F = \varphi^{-1}: G \longrightarrow H$; F is even lower semicontinuous and has a closed graph. Nevertheless F need not be upper-Baire (it is upper-Baire provided H is analytic; see Lemma 4). Take for instance $H = R \times R_d$ and $H_0 = \{0\} \times R_d$, where R_d denotes R (the reals) endowed with the discrete topology. Choose a set A in R which has not the Baire property and put $K = \{(x,x) \in H : x \in A\}$; K is closed and $F^{-1}(K) = A$.

Now we are in a position to derive the Souslin-graph theorem.

<u>Theorem 7</u>. Let $F: G \longrightarrow H$ be a multihomomorphism, where H is an analytic group. Assume that

(i) $F^{-1}(H)$ is of second category in G; or

(ii) $F^{-1}(H) = G$ and the topology on G is inductively generated by homomorphisms $h_\alpha : G_\alpha \longrightarrow G$, where $\{G_\alpha : \alpha \in A\}$ is a family of second category groups.

If Gr F is a Souslin set in $G \times H$, then F is lower semicontinuous.

<u>Proof</u>. (i) Follows from Lemmas 4, 5 and Theorem 5 (ii).

(ii) Fix any $\alpha \in A$. By Lemma 4, F is upper-Souslin; hence $F \circ h_\alpha$ is upper-Souslin, and so upper-Baire. Lemma 5 shows that $F \circ h_\alpha$ is lower-Baire. By Theorem 5 (ii), $F \circ h_\alpha$ is lower semicontinuous. Since α was arbitrary, the assertion follows.

Clearly, Theorem 7 yields the statements mentioned in the passage before Lemma 4.

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