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ON COVERINGS OF RANDOM GRAPHS
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Abstract: It is shown that almost all graphs have the property that almost all edges can be covered by edge disjoint triangles. Various generalizations of this statement are considered.

Key words: Random graph, covering.

Classification: 05C99

Many papers have dealt recently with the problem of decomposing a graph into isomorphic subgraphs. In this note we investigate related questions concerning random graphs. Let \( n \) be a positive integer; is it true that the majority of graphs with \( n \) vertices can be decomposed into edge disjoint triangles (or more generally into edge disjoint copies of a given graph \( F \)) so that only relatively few edges are left?

We prove, provided \( n \) is sufficiently large that it is so. (For the more detailed definitions concerning random graphs see [2].)

Theorem. Let \( \varepsilon \) be a positive, \( \varepsilon \leq 1 \) and \( C_j = (V, \mathcal{E}) \) a random graph with \( n \) vertices, such that each edge is present with the prescribed probability \( p \), independently of the presence or absence of any other edges. Then, with probability
tending to one (as $n \to \infty$) there exists a system $T(C)$ of edge disjoint triangles in $G$ so that all but at most $\varepsilon n^2$ edges are covered by some triangle from $T(C)$.

**Proof:** A) We can clearly suppose without loss of generality that $n = 6m + 1$ or $6m + 3$. Let $K = K_n$ be a complete graph with the vertex set $V$. From the existence of Steiner triple systems with $n$ vertices ($n \equiv 1$ or $3 \pmod{6}$) it immediately follows that there exists a covering $C_0$ of the edges of complete graph $K = K_n$ by edge disjoint triangles. Let $\pi_1, \pi_2, \ldots, \pi_N$ be independent random permutations of the vertices in $V$, $N$ will be chosen later. We assume that these permutations are also independent of the random graph $G$. (In other words, we work on a product space $\{0, 1\}^{(\binom{n}{3})} \times \pi^N$ with the product measure $P = p^{(\binom{n}{3})} \times \pi^N$ where $\pi$ is the set of all permutations of $\{1, \ldots, n\}$ each one having $\mu$-measure $1/n!$, and $P(1) = p$, $P(0) = 1 - p$.) We define the independent coverings $C_1, \ldots, C_N$ as follows: a triangle $\{v_1, v_2, v_3\}$ belongs to $C_1$ if $\{\pi_1 v_1, \pi_1 v_2, \pi_1 v_3\}$ belongs to $C_0$.

Now our algorithm goes as follows. Select all triangles in $G$ that appear in $C_1$, then all triangles appearing in $C_2$ that are edge disjoint from the ones selected before, etc. This way we cover some portion of the edges of $G$ by edge disjoint triangles, and hopefully a large portion.

Define the indicator variables

$$\chi_e = \begin{cases} 1 & \text{if } e \in C, \text{ nevertheless } e \text{ has not been covered in our procedure} \\ 0 & \text{otherwise} \end{cases}$$

and set $d = E\chi_e$, where $E$ denotes the expectation of random variable $\chi_e$. ($E\chi_e$ does not depend on $e$ because of complete
symmetry.) For the number $D$ of edges not covered we have

$$D = \sum_{e \in \binom{E}{2}} \mathcal{N}_e \left( \binom{V}{2} \right)$$

is a set of all pairs of $V$, $E_D = \binom{\binom{V}{2}}{2}$.

Now

$$P(D \sim \binom{n}{2} p) \leq E_D / E \binom{n}{2} p = \frac{d_f}{p}$$

and

$$P(\lambda_2 < \frac{1}{2} \binom{n}{2} p) = o(1)$$

(if only $\binom{n}{2} p \to \infty$), thus in order to show that $D / \lambda_2 \to 0$ it is sufficient to show that $d_f / p \to 0$.

B) Define the numbers $p_i$ recursively as follows

$$p_0 = 0$$

$$p_{k+1} = p_k + (p - p_k)^3$$

Taking $d_k = p - p_k$ we have thus

$$d_0 = p, d_{k+1} = d_k - d_k^3$$

It is easy to see that $d_k \to 0$ (actually $d_k \sim \frac{1}{\sqrt{2k}}$). Moreover, since $d_k$ is decreasing we have $0 < d_k < p - kd_k^3$ whence

$$d_k \sim \left( \frac{p}{k} \right)^{1/3}, \quad k = 1, 2, \ldots$$

Now we are going to prove

$$d < d_n \cdot \frac{9^N}{n}$$

and thus $d_f / p \to 0$ if only $\frac{9^N}{np} \to 0$ and $Np^2 \to \infty$ which holds if $p \sqrt{\log n} \to \infty$ (choose $N = \frac{1}{10 \log n}$).

C) Consider an edge $e$. Let $T_k = T_k(e)$ denote the triangle in $C_k$ that cover $e$. Start with $T_N(e)$.

In $C_{N-1}$ there are three triangles (not necessarily different) containing the edges of $T_N(e)$. In $C_{N-2}$ there are nine triangles containing the nine edges that appeared so far, etc.

Let $A = A(e)$ denote the event that the $3 + 3^2 + \ldots + 3^N =$
= \frac{3}{2}(3^n - 1) edges thus appearing are all different, and $B_k = B_k(e)$ the event that the edge $e$ is covered up to the K-th step of our procedure ($k=1,\ldots,N$).

We fix the covering $C_1,\ldots,C_N$ in such a way that $A$ holds, and randomize $C_j$. Define the conditional probability

$$P_k = P(B_k|C_1,\ldots,C_N)$$

for these fixed coverings.

For the probability $P_{k+1} - P_k$ that $e$ gets covered in exactly the $(k+1)$-th step, we obviously have

$$P_{k+1} - P_k = (p - P_k)^3, \quad P_1 = p^3$$

since the three edges of $T_k(e)$ have to be drawn in $C_j$ and should not have been covered earlier (this explains $p - P_k$), moreover, these three events are independent, for we fixed the $C$-s in $A(e)$.

Thus $P_k$, and also their mixture $P(B_k|A)$ satisfy (1), and hence are equal to $p_k$.

We have

$$d = p - P(B_N) = p - P(B_N|A)P(A) - P(B_N|\overline{A})P(\overline{A}) = p - p_NP(A) \leq$$

$$\leq p - p_N + P(\overline{A}) = d_N + P(\overline{A}).$$

Now

$$P(\overline{A}) \leq \sum_{k=1}^{N-1} \frac{3^k}{n} < 2.9^k/n < 9^k/n$$

for up to the k-th step (backwards) in the above argument C) we have $3^k$ edges altogether, and the probability that the corresponding random $3^k$ points (one step back) are all different from the $(3^k + 3)/2$ points obtained so far, is less than $2.9^k/n$. Q.E.D.
Remark. Here we outline that in our theorem triangle can be replaced by any other graph $F$. Consider a graph (configuration of edges) $F$ which $K_n$ can be covered by. An important result of R.M. Wilson [1] shows that the trivial necessary conditions for $n$ are also "asymptotically sufficient" and hence $K_n$ can be covered by edge disjoint copies of $F$ for all sufficiently large $n$ satisfying the necessary conditions.

If $F$ contains $r$ edges rather than three, then we have to change (1) to

$$p_{k+1} = p_k + (p - p_k)^r, p_0 = 0$$

which leads to

$$d_k = p - p_k \sim ((r - 1)k)^{1/r-1}$$

and also (3) to

$$d < d^*_M + c^2 N / n$$

which leads to the condition

$$p(\log n / \log r)^{1/r-1} \to \infty$$

Thus, with $p = \text{const}$ (say $1/2$), the procedure works for covering with subgraphs with $o(\log \log n)$ edges, e.g. for $o(\sqrt{\log \log n})$-gons.

For fixed $r$ we have seen that the procedure works as long as

$$p(\log n)^{1/r-1} \to \infty$$

i.e. as long as the number of edges is much larger than

$$n^2 / (\log n)^{1/r-1}.$$
\[ n^2/(\log n) \cdot \]

A good guess is, however, that even a random graph with 
\[ \omega(n)n^{3/2}, \omega(n) \to \infty \]
edges can be covered almost perfectly. This would be a strong statement and is completely beyond the power of our method.\(x\)^)

References


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\(x\) Added in proofs: Recently we have proved this conjecture.