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ON THE EQUATION $x' = f(t, x)$ IN BANACH SPACES

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Abstract: In this paper, we deal with the existence theorem for the equation $x' = f(t,x)$, where the values of a function $f(t,x)$ lie in an arbitrary Banach space. In order to obtain the existence of solutions of this equation we assume that the function $f(t,x)$ is uniformly continuous and satisfies some comparison condition involving the notion of a measure of noncompactness which is defined in an axiomatic way.

Key words: Ordinary differential equation in Banach space, measure of noncompactness, fixed point theorem of Darbo type.

Classification: 47H09, 34G20

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1. Introduction. The purpose of this paper is to prove some existence theorems for an ordinary differential equation in Banach space. We assume that the right hand side of that equation satisfies a comparison condition of Lipschitz type translated in terms of a so-called measure of noncompactness. The notion of a measure of noncompactness which we will use, was defined in an axiomatic way in the work [3] (cf. also [2]). This axiomatics is not so general as that of Sadovskii [12] but it seems to be very convenient in a lot of applications because it admits many natural realizations [3].

It is worth to mention that the notion of a measure of noncompactness was very intensively examined in the last years and it was used in many branches of nonlinear functional analysis ([3],[5],[7],[11],[12]). The application of measures of noncompactness in the theory of ordinary differential equations in Banach spaces, was at first initiated by Ambrosetti [1]. After Ambrosetti's paper, there have appeared many papers involving differential equations together with measures of noncompactness ([8],[4],[14],[12],[7]). In almost all of the mentioned papers there has been used the measure of noncompactness defined by Kuratowski [9]. Notice that Kuratowski's measure is very convenient but in several Banach spaces we do not know any convenient necessary and sufficient criteria of compactness and therefore the application of Kuratowski's measure is very difficult and even impossible. With regard to this we use measures defined in an axiomatic way, which allows us to omit the mentioned difficulties.

2. Basic notations and definitions. Let E be an arbitrary Banach space with the norm $\|\cdot\|$ and the zero element θ and let $K(x,r)$ denote the closed ball centered at x and with radius r. Denote by $\mathfrak{M}_E$ the family of all bounded and nonempty subsets of E and by $\mathfrak{N}_E$ its subfamily which contains relatively compact sets. For $X,Y \subset E$ the closure, convex closure and linear combination of these sets will be denoted by $\overline{X}$, Conv X, $\alpha X + \beta Y$, respectively.

Definition [3]. A function $\mu: \mathfrak{M}_E \to \langle 0,+\infty)$ will be called a measure of noncompactness if it satisfies the following conditions:

1° the family $\mathcal{P} = [X \in \mathfrak{M}_E : \mu(X) = 0]$ is nonempty and $\mathcal{P} \subset \mathfrak{N}_E$,

2° $X \subset Y \Longrightarrow \mu(X) \leq \mu(Y)$,

3° $\mu(\bar{X}) = \mu(\mathrm{Conv}\ X) = \mu(X)$,

4° $\mu(\lambda X + (1-\lambda)Y) \leq \lambda\mu(X) + (1-\lambda)\mu(Y)$, for $\lambda \in \in \langle 0,1 \rangle$,

5° if $X_n \in \mathfrak{M}_E$, $\bar{X}_n = X_n$, $X_{n+1} \subset X_n$, $n = 1,2,\ldots$ and if $\lim_{n\to\infty} \mu(X_n) = 0$ then $X_\infty = \bigcap_{n=1}^{\infty} X_n \neq \emptyset$.

The family $\mathcal{P}$ defined in 1° is said to be the kernel of the measure μ and it is denoted by $\ker \mu$. It may be shown that the family $(\ker \mu)^c = [X \in \ker \mu : X = \bar{X}]$ forms a closed subspace of the space $\mathfrak{M}_E^c = [X \in \mathfrak{M}_E : X = \bar{X}]$ with respect to the topology generated by Hausdorff metric [3].

In the sequel, we will use the following modified version of the fixed point theorem of Darbo type ([2],[3], cf. also [6]).

<u>Theorem 1</u>. Let $C \in \mathfrak{M}_E$, Conv $C = C$ and let $T: C \longrightarrow C$ be a continuous transformation such that $TX \in \mathfrak{M}_E$ for any $X \in \mathfrak{M}_E$. If there exists a constant $k \in \langle 0,1)$ such that

$$\mu(TX) \leq k\,\mu(X),$$

then T has at least one fixed point which belongs to $\ker \mu$. Moreover, the set $\mathrm{Fix}T = [x \in C : Tx = x]$ belongs to $\ker \mu$.

Further, for any measure μ defined in the space E, we will denote

$$E_\mu = [x \in E : \{x\} \in \ker \mu].$$

Obviously E_μ is a closed and convex subset of the space E. In the case when μ is a sublinear measure, i.e. if it satisfies, in addition, the following two conditions

$$\mu(X + Y) \leq \mu(X) + \mu(Y),$$

$$\mu(\lambda X) = |\lambda| \mu(X), \ \lambda \in R,$$

then E_μ forms a closed linear subspace of E.

Next, by $C(\langle 0,T\rangle, E)$ or shortly by C we will denote the space of all continuous functions acting from the interval $\langle 0,T\rangle$ into E, with the usual maximum norm.

For an arbitrary $X \in \mathfrak{M}_C$ and $\varepsilon > 0$ we put:

$$\omega(X,\varepsilon) = \sup_{x \in X} \{\sup [\|x(t) - x(s)\|_E : t,s \in \langle 0,T\rangle, |t-s| \leq \varepsilon]\},$$

$$\omega_0(X) = \lim_{\varepsilon \to 0} \omega(X,\varepsilon),$$

$$X(t) = [x(t) : x \in X],$$

$$M(X) = \sup [\mu_E(X(t)) : t \in \langle 0,T\rangle] .$$

Finally, let us define

$$\mu(X) = \omega_0(X) + M(X).$$

This function is the measure of noncompactness in C with the kernel $\mathcal{P}_C$ consisting of all equicontinuous sets X such that $X(t) \in \ker \mu_E$ for any $t \in \langle 0,T\rangle$ [3].

Notice that the function M(X) is the measure of noncompactness on the family $\mathfrak{M}_C^{eq}$ of equicontinuous sets.

3. Some properties of measures of noncompactness. Let μ be an arbitrary measure of noncompactness in the space E. For $X \in \mathfrak{M}_E$ let us denote

$$\|X\| = \sup [\|x\| : x \in X] .$$

Now we prove a few lemmas describing some properties of a measure μ. These lemmas generalize some results given in [3],[2].

Lemma 1. If $\|X\| \leq 1$, then

$$\mu(X + Y) \leq \mu(Y) + \|X\| \mu(K(Y,1)),$$

where

$$K(Y,1) = \bigcup_{y \in Y} K(y,1).$$

For the proof see [2].

Lemma 2. Let $\{x_0\} \in \ker \mu$. Then

$$\mu(x_0 + tX) \leq t\, \mu(x_0 + X)$$

for $t \in \langle 0,1 \rangle$.

Proof. Using the axiomatics of a measure we have

$$\mu(x_0 + tX) = \mu(tx_0 + (1-t)x_0 + tX) = \mu((1-t)x_0 + t(x_0 + X)) \leq$$
$$\leq (1-t)\,\mu(\{x_0\}) + t\,\mu(x_0 + X) = t\,\mu(x_0 + X),$$

which proves our lemma.

Lemma 3. Let $t_1, t_2, \dots, t_n$ be given nonnegative reals such that $\sum_{i=1}^{n} t_i \leq 1$ and let $\{x_0\} \in \ker \mu$. Then

$$\mu(x_0 + \sum_{i=1}^{n} t_i X_i) \leq \sum_{i=1}^{n} t_i\, \mu(x_0 + X_i).$$

Proof. If $\sum_{i=1}^{n} t_i = 0$, then the inequality is obvious. Let $\sum_{i=1}^{n} t_i > 0$. Denoting $\lambda_i = t_i / \sum_{k=1}^{n} t_k$, with respect to Lemma 2 and our definition and using the fact that $\sum_{i=1}^{n} \lambda_i = 1$, we have

$$\mu(x_0 + \sum_{i=1}^{n} t_i X_i) = \mu(x_0 + (\sum_{i=1}^{n} t_i)(\lambda_1 X_1 + \lambda_2 X_2 + \dots$$
$$\dots + \lambda_n X_n)) \leq (\sum_{i=1}^{n} t_i)\, \mu(x_0 + \sum_{i=1}^{n} \lambda_i X_i) =$$
$$= (\sum_{i=1}^{n} t_i)\, \mu(\sum_{i=1}^{n} \lambda_i (x_0 + X_i)) \leq$$
$$\leq (\sum_{i=1}^{n} t_i) \sum_{i=1}^{n} \lambda_i\, \mu(x_0 + X_i) = \sum_{i=1}^{n} t_i\, \mu(x_0 + X_i),$$

and the proof is complete.

Lemma 4 [3]. Each measure of noncompactness is locally Lipschitzian (hence continuous) with respect to the Hausdorff metric.

Now, let us fix a measure of noncompactness in the space

E and let $\{x_0\} \in \ker \mu$. For $X \in \mathfrak{M}_C$ we denote

$$\int_0^t X(s)ds = [\int_0^t x(s)ds : x \in X].$$

We prove some generalization of Goebel and Rzymowski - Lemma [8].

Lemma 5. If $X \in \mathfrak{M}_C^{eq}$ then for any $t \in \langle 0, \min\{1,T\}\rangle$ the following inequality holds:

$$\mu(x_0 + \int_0^t X(s)ds) \leq \int_0^t \mu(x_0 + X(s))ds.$$

Proof. Notice first that in view of Lemma 4 the function $t \to \mu(x_0 + X(t))$ is continuous, also integrable. Further, let us take an arbitrary $\varepsilon \in (0,1)$. In virtue of equicontinuity we can choose points $0 = t_0 \leq \xi_1 \leq t_1 \leq \xi_2 \leq \ldots \leq \xi_n \leq t_n = t$ so densely in $\langle 0,t\rangle$ that for all $x \in X$

$$\left\| \int_0^t x(s)ds - \sum_{i=1}^n x(\xi_i)(t_i - t_{i-1}) \right\| \leq \varepsilon .$$

Thus we get

$$x_0 + \int_0^t X(s)ds \subset [\int_0^t x(s)ds - \sum_{i=1}^n x(\xi_i)(t_i - t_{i-1}) : x \in X] +$$

$$+ [x_0 + \sum_{i=1}^n x(\xi_i)(t_i - t_{i-1}) : x \in X] = A + B.$$

Hence in view of Lemma 1 we obtain

$$\mu(A + B) \leq \mu(B) + \|A\| \mu(K(B,1)) \leq \varepsilon \mu(K(B,1)) +$$

$$+ \mu[x_0 + \sum_{i=1}^n x(\xi_i)(t_i - t_{i-1}) : x \in X].$$

Further, applying Lemma 3 we have

$$\mu(x_0 + \int_0^t X(s)ds) \leq \mu(x_0 + \sum_{i=1}^n X(\xi_i)(t_i - t_{i-1})) +$$

$$+ \varepsilon \mu(K(B,1)) \leq \sum_{i=1}^n (t_i - t_{i-1}) \mu(x_0 + X(\xi_i)) + \varepsilon \mu(K(B,1)).$$

Finally, densifying the partition of the interval $\langle 0,t\rangle$ and taking into account that the number ε is arbitrary, we obtain

$$\mu(x_0 + \int_0^t X(s)ds) \le \int_0^t \mu(x_0 + X(s))ds,$$

which completes the proof.

4. <u>Existence of solutions of ordinary differential equations</u>. Let us consider the ordinary differential equation

(1) $$x' = f(t,x)$$

with the initial condition

(2) $$x(0) = x_0.$$

We shall assume that f is defined on $\langle 0,T\rangle \times K(x_0,r)$, uniformly continuous and bounded, $\| f(t,x) \| \le A$. Moreover, we will assume that for all $X \in \mathfrak{M}_C$ the following inequality holds:

(3) $\mu(x_0 + f(t,X)) \le p(t)\,\mu(X)$, for almost all $t \in \langle 0,T\rangle$,

where μ is a given measure of noncompactness in the space E such that $\{x_0\} \in \ker \mu$ and p(t) is a Lebesgue integrable function on $\langle 0,T\rangle$.

Notice that if we denote $g(t,x) = x_0 + f(t,x)$ then for any $x \in E_\mu$ in view of (3) we obtain

$$\mu(g(t,x)) = \mu(x_0 + f(t,x)) \le p(t)\,\mu(\{x\}) = 0$$

for almost all $t \in \langle 0,T\rangle$, so that in virtue of continuity of f we have that $g: \langle 0,T\rangle \times E_\mu \longrightarrow E_\mu$. Particulary, $x_0 + f(0,x_0) \in E_\mu$ and in view of Lemma 2 we can easily deduce that the tangent segment $[x_0 + tf(0,x_0) : t \in \langle 0,1\rangle]$ is a subset of E_μ.

Now we prove the following theorem.

<u>Theorem 2</u>. Under the above assumptions, if $AT \le r$, $T \le 1$, then the equation (1) has at least one solution satisfying the condition (2). Moreover, all solutions of the problem (1)-(2) are such that $x(t) \in E_\mu$, for all $t \in \langle 0,T\rangle$.

<u>Proof</u>. Let $X_0 \subset C = C(\langle 0,T\rangle, E)$ be the set of all functions x such that $x(0) = x_0$ and $\|x(t) - x(s)\|_E \leq A|t-s|$. Notice that X_0 is closed, bounded, convex and equicontinuous. Furthermore, the transformation

$$(Fx)(t) = x_0 + \int_0^t f(s,x(s))ds$$

maps continuously X_0 into itself. Thus our problem is equivalent to the existence of a fixed point of F.

Fix a number $\varkappa > 1$ and for any $X \in \mathfrak{M}_C^{eq}$ let us put

$$\mu_\varkappa(X) = \sup \left[\mu(X(t)) \exp\left(-\varkappa \int_0^t p(s)ds\right) : t \in \langle 0,T\rangle\right].$$

One can show that $\mu_\varkappa(X)$ satisfies the axioms of measure of noncompactness on the family $\mathfrak{M}_C^{eq}$.

Then, in view of Lemma 5 we have

$$\mu((FX)(t)) = \mu\left(x_0 + \int_0^t f(s,X(s))ds\right) \leq \int_0^t \mu(x_0 +$$
$$+ f(s,X(s))ds \leq \int_0^t p(s)\,\mu(X(s))ds \leq \mu_\varkappa(X)\int_0^t p(s)$$
$$\exp\left(\varkappa \int_0^s p(\tau)d\tau\right)ds \leq \exp\left(\varkappa \int_0^t p(s)ds\right) \frac{1}{\varkappa}\,\mu_\varkappa(X).$$

After dividing both sides by $\exp(\varkappa \int_0^t p(s)ds)$ and taking supremum on the left hand we get

$$\mu_\varkappa(FX) \leq \frac{1}{\varkappa}\,\mu_\varkappa(X).$$

Thus, applying Theorem 1, we complete the proof.

<u>Remark</u>. It is worth to mention that in the case $x_0 = \Theta$, $\{\Theta\} \in \ker \mu$, the comparison condition (3) has the form

$$\mu(f(t,X)) \leq p(t)\,\mu(X) \tag{4}$$

(cf. [2]). Moreover, if μ is a sublinear measure, then the condition (3) is equivalent to the condition (4). Indeed, we have

$$\mu(x_0 + f(t,X)) \leq \mu(\{x_0\}) + \mu(f(t,X)) = \mu(f(t,X)).$$

On the other hand,

$$\mu(f(t,X)) = \mu(x_0 + f(t,X) + \{-x_0\}) \leq \mu(x_0 + f(t,X)) +$$
$$+ \mu(\{-x_0\}) = \mu(x_0 + f(t,X)),$$

so that

$$\mu(f(t,X)) = \mu(x_0 + f(t,X)).$$

Now we give a few examples.

Example 1. Let us consider the infinite system of differential equations

(5) $$x_n' = a_n(t)x_n + f_n(x_n, x_{n+1}, \dots), \ n = 1,2,\dots,$$

with the initial conditions

(6) $$x_n(0) = x_0^n, \ n = 1,2,\dots .$$

We will assume that there exists $\lim_{n\to\infty} x_0^n = a$. Moreover, we assume that:

(i) $a_n: \langle 0,T\rangle \to R$ are continuous functions such that the sequence $a_n(t)$ converges uniformly on the interval $\langle 0,T\rangle$ to the function which vanishes identically,

(ii) there exists a sequence of real nonnegative numbers a_n such that $\lim_{n\to\infty} a_n = 0$ and $|f_n(x_n, x_{n+1}, x_{n+2}, \dots)| \leq a_n$ for $n = 1,2,\dots$ and for all $x = (x_1, x_2, \dots) \in l^\infty$,

(iii) the function $f = (f_1, f_2, \dots)$ transforms the space l^∞ into itself and is continuous.

Under the above hypotheses the initial value problem (5)--(6) has at least one solution $x(t)$ such that $x(t) \in l^\infty$ for any $t \in \langle 0,T\rangle$ and $\lim_{n\to\infty} x_n(t) = a$ uniformly with respect to $t \in \langle 0,T\rangle$, where $T \leq 1$.

For the proof let us take into account the measure of noncompactness in the space l^{∞} defined as follows:

$$\mu(X) = \lim_{n \to \infty} \sup [\sup_{x \in X} |x_n - a|]$$

(cf. [3]). The kernel ker μ of this measure is the family of all bounded subsets of the space l^{∞} consisting of sequences which converge to a with the same "rate".

Now, for $X \in \mathfrak{M}_{l^{\infty}}$ we have

$$\mu(x_0 + f(t,X)) = \lim_{n \to \infty} \sup [\sup_{x \in X} |x_0^n + a_n(t)x_n + f_n(x_n,x_{n+1},\dots$$
$$\dots) - a|] \leq \lim_{n \to \infty} \sup [\sup_{x \in X} [|a_n(t)|\,|x_n| + |f_n(x_n,x_{n+1},\dots) +$$
$$+ x_0^n - a|]] \leq \lim_{n \to \infty} \sup [\sup_{x \in X} [p(t)|x_n - a| + |a_n(t)|\,|a| +$$
$$+ |f_n(x_n,x_{n+1},\dots)| + |x_0^n - a|]],$$

where $p(t) = \sup [|a_n(t)| : n = 1,2,\dots]$, $t \in \langle 0,T \rangle$. Hence we obtain

$$\mu(x_0 + f(t,X)) \leq p(t)\, \mu(X),$$

which proves our assertion.

<u>Example 2</u>. Now, let us take the infinite system of differential equations of the form

$$x_n' = p_n(t)x_n + f_n(x_1,x_2,\dots), \quad t \in \langle 0,T \rangle \tag{7}$$

with the initial condition

$$x_n(0) = a_n, \quad n = 1,2,\dots, \tag{8}$$

where a_n is a sequence of nonnegative reals converging to zero. We assume that the functions $f_n : l^{\infty} \to R$ are such that there is a sequence b_n converging to zero and $|f_n(x)| \leq b_n$ for $x \in l^{\infty}$, and besides, the function $f = (f_1,f_2,\dots) : l^{\infty} \to l^{\infty}$ is uniformly continuous. Further, let us assume that $p_n : \langle 0,T \rangle \to R$ are continuous functions and such that $|p_n(t)| \leq p(t)$, $t \in \langle 0,T \rangle$, where $p : \langle 0,T \rangle \to R$ is some continuous function.

In what follows, let us consider the measure of noncompactness on the space l^{∞} defined by the formula

$$\mu(X) = \lim_{n\to\infty}[\sup_{x\in X}[\sup[|x_k|:k\geq n]]].$$

One can check that $\mu(X)$ is the sublinear measure such that its kernel is a collection of all bounded sets in the space l^{∞} consisting of sequences converging to zero with the same rate [3]. Notice that if we denote $x_0 = (a_1,a_2,\ldots)$ then $\{x_0\}\in \ker\mu$.

Now we show that the problem (7)-(8) has at least one solution in the space l^{∞} provided $T\leq 1$ and the above assumptions are satisfied.

Note at first that for $X\in \mathfrak{M}_{l^{\infty}}$ we get

$$\mu(f(t,X))\leq \lim_{n\to\infty}[\sup_{x\in X}[\sup[|p_k(t)||x_k| + |f_k(x_1,x_2,\ldots)|:$$
$$:k\geq n]]] \leq \lim_{n\to\infty}[\sup_{x\in X}[\sup[p(t)|x_k| + b_k:k\geq n]]] = p(t)\mu(X).$$

Hence, in view of Remark made after Theorem 2 we conclude that the problem (7)-(8) admits at least one solution $x(t) = (x_1(t),x_2(t),\ldots)$ in the space l^{∞}, where the sequences $x_n(t)$ converge to zero on the interval $\langle 0,T\rangle$ with the same rate.

<u>Example 3</u>. Now we pay our attention to the case which is not covered by Theorem 2. Namely, assume that μ is an arbitrary measure of noncompactness on the space E and $x_0\in E_{\mu}$. Further, let us suppose $f:\langle 0,T\rangle\times K(x_0,r)\to K(x_0,r)$ is a given function which is uniformly continuous and such that

$$(9)\qquad \mu(x_0 + f(t,X))\leq\frac{\mu(X)}{t^2} \text{ for } t\in (0,T\rangle,\ X\subset K(x_0,r)$$

and

$$(10)\qquad \mu(x_0 + f(t,X)) = o(e^{-\frac{1}{t}}/t^2) \text{ as } t\to 0+, \text{ uniformly with}$$

respect to $X\subset K(x_0,r)$.

Then we have the following theorem (cf. [15]).

<u>Theorem 3</u>. Let $T \leq 1$. Then the equation (1) has at least one solution x such that the condition (2) is satisfied. Apart from that $x(t) \in E_\mu$ for all $t \in \langle 0,T\rangle$.

<u>Proof</u>. Similarly as in the proof of Theorem 2 we consider the set X_0 defined there and reduce our problem to the existence of fixed points of the transformation $\mathcal{F}$ defined on the set X_0 with help of the formula

$$(\mathcal{F}x)(t) = x_0 + \int_0^t f(s,x(s))ds.$$

Next, consider the sets $X_{i+1} = \operatorname{Conv} \mathcal{F} X_i$, $i = 0,1,2,\ldots$. All these sets are of the same type as X_0 and $X_{i+1} \subset X_i$. Let us put

$$u_n(t) = \mu(X_n(t)),\ t \in \langle 0,T\rangle .$$

We have that $0 \leq u_{n+1}(t) \leq u_n(t)$. Moreover, in view of the inequality

$$|\mu(X(t)) - \mu(X(s))| \leq a\,\omega(X, |t-s|)$$

(cf. [3]), all these functions are equicontinuous. Consequently, the sequence $u_n(t)$ converges uniformly to a function $u_\infty(t) = \lim_{n \to \infty} u_n(t)$. Observe now that from (9) and Lemma 5 follows

$$\begin{aligned} u_{n+1}(t) &= \mu(x_0 + \int_0^t f(s,X_n(s))ds) \leq \int_0^t \mu(x_0 + \\ &+ f(s,X_n(s)))ds \leq \int_0^t \frac{\mu(X_n(s))}{s^2}ds = \int_0^t \frac{u_n(s)}{s^2}ds . \end{aligned} \tag{11}$$

Fixing an arbitrary $\varepsilon > 0$ and using (10) we deduce that there exists $\delta > 0$ such that

$$\mu(x_0 + f(t,X)) \leq \varepsilon e^{-\frac{1}{t}}/t^2, \text{ for } t \in (0,\delta\rangle ,\ X \subset K(x_0,r).$$

Hence we get

$$u_n(t) = \mu(x_0 + \int_0^t f(s,X_{n-1}(s))ds) \leq \int_0^t \mu(x_0 + f(s,X_{n-1}(s)))ds \leq \varepsilon \int_0^t (e^{-\frac{1}{s}}/s^2)ds = \varepsilon e^{-\frac{1}{t}}$$

for $t \in (0,\delta\rangle$, which implies that $u_n(t) = o(e^{-\frac{1}{t}})$ as $t \to 0+$ and consequently

$$u_\infty(t) = o(e^{-\frac{1}{t}}), \text{ as } t \to 0+. \tag{12}$$

From the above facts we conclude that the functions $t \to \frac{u_n(t)}{t^2}$ are integrable. Hence and from (11) we obtain

$$u_\infty(t) \leq \int_0^t \frac{u_\infty(s)}{s^2}\, ds.$$

The above inequality and (12) imply, via Roger's Lemma [10], that $u_\infty(t) \equiv 0$. Finally observe that

$$\lim_{n \to \infty} \{\max [u_n(t) : t \in \langle 0,T\rangle]\} = 0,$$

hence we deduce that the set $X_\infty = \bigcap_{n=1}^{\infty} X_n$ is nonempty, convex, closed and $X_\infty \in \ker \mu$. Now we apply the classical Schauder's fixed point theorem, which completes the proof.

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