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PRERADICALS AND GENERALIZATIONS OF QF-3' MODULES II.

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Abstract: The concept of dQF-3'' modules is dual to that of QF-3'' which was introduced in [18] and generalizes the concept of pseudoprojective module, in the literature (see [1], [4], [14]) also denoted as the dQF-3' module. In the following dQF-3'' modules are characterized in terms of preradicals. Some results on dQF-3'' modules and preradicals connected with dQF-3'' modules are obtained.

Key words: G-cohereditary preradicals, G-hereditary preradicals, dQF-3'' modules.

Classification: 16A63, 16A50

All the rings considered below will be associative with unit and R-mod will denote the category of all unitary left R-modules.

A preradical r for R-mod is any subfunctor of the identity functor. For the basic notions from the theory of preradicals we refer to the first part of this article (see [18]).

The class of all r-torsion (r-torsionfree) modules will be denoted by $\mathcal{T}_r$ ($\mathcal{F}_r$).

We say that a preradical r

- is superhereditary if it is hereditary and $\mathcal{T}_r$ is closed under direct products,
- has FCgSP if r(M) is a direct summand in M for every fini-

tely cogenerated module M.

The identity functor will be denoted by id. For a module Q let us define an idempotent preradical $p_{\{Q\}}$ by $p_{\{Q\}}(M) = \Sigma \operatorname{Im} f$, where f runs over all $f \in \operatorname{Hom}_R(Q,M)$, $M \in R$-mod. The idempotent core (radical closure) of a preradical r will be denoted by $\overline{r}$, $(\breve{r})$. $\bigcap_{i \in I} r_i$ $(\sum_{i \in I} r_i)$ denotes the intersection (sum) of a family of preradicals $\{r_i; i \in I\}$.

For a submodule A of a module B and a preradical r let us define $C_r(A:B)$ by $C_r(A:B)/A = r(B/A)$. If r, s are preradicals then $(r \triangle s)$ is a preradical defined by $(r \triangle s)(M) = C_s(r(M):M)$, $M \in R$-mod; $r \leq s$ means $r(M) \subseteq s(M)$ for every $M \in R$-mod.

The socle will be denoted by Soc, the injective hull (projective cover) of a module Q by E(Q) (C(Q)).

A module M is called

- finitely coembedded if there is a finitely cogenerated module N and an epimorphism $f: N \to M$,
- cocyclic if it is an essential extension of a simple module,
- cofaithful if every injective module is $p_{\{M\}}$-torsion.

A ring R is called

- left perfect if every left R-module has a projective cover,
- left V-ring if every simple left R-module is injective.

A preradical r is said to be

- an 1-radical if $M/r(M) \in \mathcal{F}_r$ for every finitely cogenerated module M,
- a 2-radical if $M/r(M) \in \mathcal{T}_r$ for every finitely coembedded module M,

- G-cohereditary if $r(B/A) = (r(B) + A)/A$, whenever $A \subseteq B$, B finitely cogenerated,
- G_1-cohereditary if for every $Q \in \mathcal{T}_r$ there is a projective presentation $0 \to K \hookrightarrow P \to Q \to 0$ of Q such that for every $X \subseteq P$ with P/X finitely cogenerated $K + C_r(X:P) = P$,
- G-hereditary if $r(M) = \bigcap C_r(X:M)$, where X runs over all submodules X of M with M/X finitely cogenerated, $M \in R$-mod.

For a preradical r let us define preradicals $(Gch)(r)$ and $(Gh)(r)$ as follows:

$(Gch)(r)(Q) = r(Q) \cap (\bigcap g(C_r(X:P)))$, where $0 \to K \hookrightarrow P \xrightarrow{g} Q \to 0$ is a projective presentation of Q, X runs over all submodules of P with P/X finitely cogenerated, $Q \in R$-mod, $(Gh)(r)(Q) = \bigcap C_r(X:Q)$, where X runs over all submodules of Q with Q/X finitely cogenerated, $Q \in R$-mod.

Proposition 1

(i) Every G-cohereditary preradical is G_1-cohereditary.

(ii) Every G_1-cohereditary idempotent preradical is G-cohereditary.

(iii) $(Gch)(r)$ is a preradical and $(Gch)(r) \le r$. Moreover if R is left perfect then $(Gch)(r)$ is G_1-cohereditary.

(iv) If $s \le r$, s G-cohereditary then $s \le (Gch)(r)$.

(v) $(Gch)(r)(Q)$ does not depend on particular choice of a projective presentation of Q.

(vi) $\overline{(Gch)(r)}$ is the largest G-cohereditary idempotent preradical contained in r provided that R is left perfect.

(vii) $(Gh)(r)$ is a G-hereditary preradical and $r \le (Gh)(r)$.

(viii) If $r \leq s$, s G-hereditary then $(Gh)(r) \leq s$.

(ix) $(Gh)(r)$ is the least G-hereditary preradical containing r.

(x) $(Gh)(r)(Q) = r(Q)$ for every finitely cogenerated module Q.

(xi) $(Gch)(r)(Q) = r(Q)$ for every projective module Q.

(xii) Every cohereditary and every superhereditary preradical is G-hereditary.

(xiii) If $\{r_i; i \in I\}$ is a family of G-cohereditary preradicals then $\sum_{i \in I} r_i$ is G-cohereditary.

(xiv) If r is a preradical then $\sum \{s; s \leq r$, s G-cohereditary (idempotent) preradical$\}$ is the largest G-cohereditary (idempotent) preradical contained in r.

(xv) If $\{r_i; i \in I\}$ is a family of G-hereditary preradicals then $\bigcap_{i \in I} r_i$ is G-hereditary.

(xvi) If r is a preradical then $\bigcap \{s; r \leq s$, s G-hereditary (pre)-radical$\}$ is the least G-hereditary (pre)-radical containing r.

(xvii) If r is G-cohereditary then $\overline{r}$ is so provided that R is left perfect.

(xviii) If r is G-cohereditary then $\tilde{r}$ is so.

Proof. (i) Let $0 \to K \hookrightarrow P \to Q \to 0$ be a projective presentation of an r-torsion module Q. If r is G-cohereditary, $X \subseteq P$ such that P/X is finitely cogenerated then $r((P/X)/((K+X)/X)) = (r(P/X) + ((K+X)/X))/((K+X)/X)$ and hence $K + C_r(X:P) = P$ since $Q \in \mathcal{T}_r$.

(ii) Let r be a G_1-cohereditary idempotent preradical, B be a finitely cogenerated module and $0 \to K \hookrightarrow P \xrightarrow{\varphi} r(B/A) \to$ $\to 0$ be a projective presentation of r(B/A) with the desired property. Consider the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \hookrightarrow & P & \xrightarrow{g} & r(B/A) & \longrightarrow & 0 \\ & & & & & {\scriptstyle f}\searrow & \uparrow{\scriptstyle \pi} & & \\ & & & & & & C_r(A:B), & & \end{array}$$

where π is the natural epimorphism. Then P/Ker f is finitely cogenerated and hence $K + C_r(\mathrm{Ker}\ f:P) = P$ since r is idempotent. Thus $r(B/A) = g(P) = g(K+C_r(\mathrm{Ker}\ f:P)) \subseteq \pi(r(f(P))) \subseteq$ $\subseteq \pi(r(B)) = (r(B)+A)/A$.
The remaining assertions are clear.

<u>Proposition 2</u>. Let r be an idempotent preradical. Then the following are equivalent:

(i) r is an 1-radical (2-radical),

(ii) if $0 \to A \to B \to C \to 0$ is exact, B finitely cogenerated (coembedded), $A, C \in \mathcal{T}_r$ then $B \in \mathcal{T}_r$.

<u>Proof</u>. (i) implies (ii). It follows from the fact that for an idempotent 1-radical (2-radical) and finitely cogenerated (coembedded) module T $T \in \mathcal{T}_r$ if and only if $\mathrm{Hom}_R(T,F) =$ $= 0$ for every $F \in \mathcal{F}_r$.

(ii) implies (i). Consider the exact sequence
$0 \to r(B) \hookrightarrow (r\Delta r)(B) \to (r\Delta r)(B)/r(B) \to 0$, where B is finitely cogenerated (coembedded). Then $(r\Delta r)(B) \in \mathcal{T}_r$ and consequently $B/r(B) \in \mathcal{F}_r$.

<u>Proposition 3</u>. For a preradical r the following are equivalent:

(i) r is G-cohereditary,

(ii) $r(B/A) = (r(B) + A)/A$, whenever $A \subseteq B$, B finitely coembedded,

(iii) if $B/r(B) \longrightarrow A$ is an epimorphism / A cocyclic /, and B finitely cogenerated (coembedded) then $A \in \mathcal{F}_r$,

(iv) a) r is a 1-radical (2-radical) and

b) whenever $A \subseteq B$, $B \in \mathcal{F}_r$ / B/A cocyclic /, B finitely coembedded then $B/A \in \mathcal{F}_r$.

Proof. Easy.

Proposition 4. The following are equivalent for a preradical r

(i) r is G_1-cohereditary,

(ii) for every $Q \in \mathcal{T}_r$ there is a projective presentation $0 \longrightarrow K \hookrightarrow P \longrightarrow Q \longrightarrow 0$ of Q such that for every $X \subseteq P$ with P/X finitely coembedded $K + C_r(X:P) = P$.

Proof. Obvious.

Proposition 5. Let r be a preradical. Then

(i) r is G-cohereditary if and only if (Gh)(r) is G-cohereditary,

(ii) $\bar{r}$ is G-cohereditary if and only if $\overline{(Gh)(r)}$ is G-cohereditary,

(iii) if (Gh)(r) is cohereditary then r is G-cohereditary,

(iv) if r is idempotent and $\overline{(Gh)(r)}$ is cohereditary then r is G-cohereditary,

(v) if R is a left perfect ring and r is G-cohereditary then $\overline{(Gh)(r)}$ is cohereditary.

Proof. (i)-(iv) are obvious.

(v) Let R be a left perfect ring and r be a G-cohereditary preradical. If $Q \in R$-mod, $Q \in \mathcal{T}_{(Gh)(r)}$, $0 \longrightarrow K \hookrightarrow P \longrightarrow Q \longrightarrow 0$ is a projective cover of Q and $X \subseteq P$ with P/X finitely cogenerated then $P = c_{(Gh)(r)}((X+K):P) = c_{(Gh)(r)}(X:P) + K = c_r(X:P) + K$ since $(Gh)(r)$ is G-cohereditary. Hence $c_r(X:P) = P$ and consequently $(Gh)(r)(P) = P$ which yields $\overline{(Gh)(r)}$ is cohereditary.

Corollary 6. An idempotent G-hereditary preradical in a left perfect ring is G-cohereditary if and only if it is cohereditary.

Proposition 7. Let r be an idempotent G-cohereditary preradical for a left perfect ring R. Then there is a projective (Gh)(r)-torsion module P such that $r(N) = p_{\{P\}}(N)$ for every finitely coembedded module N.

Proof. From Proposition 5 and [3], Theorem 4.7 it follows that there is a projective (Gh)(r)-torsion module P such that $\overline{(Gh)(r)} = p_{\{P\}}$. Hence $r(N) = p_{\{P\}}(N)$ for every finitely coembedded module N.

A left R-module Q is called

- dQF-3'' if the idempotent preradical $p_{\{Q\}}$ is G-cohereditary,
- r dQF-3'' if the idempotent radical $\widehat{p_{\{Q\}}}$ is G-cohereditary.

Proposition 8. Let $Q \in$ R-mod. Then the following are equivalent:

(i) Q is dQF-3'',

(ii) there is a projective presentation $0 \longrightarrow K \hookrightarrow P \longrightarrow Q \longrightarrow 0$ of Q such that $K + c_{p_{\{Q\}}}(X:P) = P$ for every $X \subseteq P$ with P/X finitely cogenerated (coembedded),

(iii) a) $Hom_R(Q, X/p_{\{Q\}}(X)) = 0$ for every finitely cogenerated (coembedded) module X and

b) if $A \subseteq B$, $\mathrm{Hom}_R(Q,B) = 0$ / B/A cocyclic / and B finitely coembedded then $\mathrm{Hom}_R(Q,B/A) = 0$,

(iv) a) if $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is exact, B finitely cogenerated (coembedded), $A \in \mathcal{T}_{p_{\{Q\}}}$ and $C \in \mathcal{T}_{p_{\{Q\}}}$ then $B \in \in \mathcal{T}_{p_{\{Q\}}}$ and

b) if $A \subseteq B$, $\mathrm{Hom}_R(Q,B) = 0$ / B/A cocylic / and B finitely coembedded then $\mathrm{Hom}_R(Q,B/A) = 0$,

(v) for every epimorphism $h:B \longrightarrow A$, where B is finitely cogenerated (coembedded), for every non-zero homomorphism $f: :Q \longrightarrow A$ there are homomorphisms $k:Q \longrightarrow Q/\mathrm{Ker}\ f$ and $g:Q \longrightarrow B$ with $0 \neq h \circ g = \bar{f} \circ k$ / $\bar{f}$ is induced by f /,

(vi) for every epimorphism $h:B \longrightarrow C$, where C is cocylic, B is finitely cogenerated (coembedded), for every nonzero homomorphism $f:Q \longrightarrow C$ there are homomorphisms $k:Q \longrightarrow Q/\mathrm{Ker}\ f$ and $g:Q \longrightarrow B$ with $0 \neq h \circ g = \bar{f} \circ k$ / $\bar{f}$ is induced by f /,

(vii) if $f:B \longrightarrow A$ is an epimorphism / A is cocylic /, B is finitely cogenerated (coembedded) and $\mathrm{Hom}_R(Q,A) \neq 0$ then there is a homomorphism $g:Q \longrightarrow B$ with $\mathrm{Im}\ g \nsubseteq \mathrm{Ker}\ f$.

Moreover, if Q has a projective cover then the conditions (i)-(vii) are equivalent to

(viii) $p_{\{Q\}}(C(Q)/X) = C(Q)/X$ for every $X \subseteq C(Q)$ with $C(Q)/X$ finitely cogenerated (coembedded),

(ix) if $X \subseteq C(Q)$ such that $C(Q)/X$ is finitely cogenerated (coembedded) then $C(Q)/X$ is isomorphic to a factormodule of a direct sum of copies of Q,

(x) $(Gh)(p_{\{Q\}}) = p_{\{C(Q)\}}$,

(xi) $(Gh)(p_{\{Q\}})$ is cohereditary,

(xii) $p_{\{Q\}}(X) = p_{\{C(Q)\}}(X)$ for every finitely cogenerated (coembedded) module X,

(xiii) $(Gh)(p_{\{Q\}})(C(Q)) = C(Q)$,

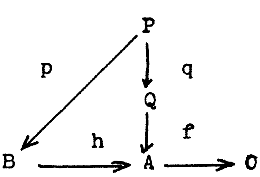
(xiv) for every finitely cogenerated (coembedded) module X $p_{\{C(Q)\}}(X) = X$ implies $p_{\{Q\}}(X) = X$,

(xv) a) if $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is exact, B finitely cogenerated (coembedded), $A \in \mathcal{T}_{p_{\{Q\}}}$ and $C \in \mathcal{T}_{p_{\{Q\}}}$ then $B \in \in \mathcal{T}_{p_{\{Q\}}}$ and

b) for every finitely coembedded module X $\mathrm{Hom}_R(Q,X) = 0$ if and only if $\mathrm{Hom}_R(C(Q),X) = 0$.

<u>Proof</u>. (ii) implies (i). Let $\mathcal{A}$ denote the class of all $N \in R$-mod for which there is a projective presentation $0 \longrightarrow L \hookrightarrow M \longrightarrow N \longrightarrow 0$ with $L + C_{p_{\{Q\}}}(X:M) = M$ for every $X \subseteq M$ with M/X finitely cogenerated (coembedded). Then $Q \in \mathcal{A}$ and $\mathcal{A}$ is a cohereditary class closed under direct sums and consequently $\mathcal{T}_{p_{\{Q\}}} \subseteq \mathcal{A}$. Now it suffices to use Proposition 1 (ii).

(ii) implies (v). Consider the following commutative diagram



with exact row,

where B is finitely cogenerated, $f \neq 0$ and $0 \longrightarrow K \hookrightarrow P \xrightarrow{q} Q \longrightarrow \longrightarrow 0$ is a projective presentation of Q such that $K + C_{p_{\{Q\}}}(X:P) = P$ for every $X \subseteq P$ with P/X finitely cogenerated.

Then $P/\ker p$ is finitely cogenerated and hence $K + C_{p_{\{Q\}}}(\mathrm{Ker}\ p:P) = P$. If for every homomorphism $t:Q \longrightarrow \longrightarrow P/\mathrm{Ker}\ p$ $q(\pi^{-1}(\mathrm{Im}\ t)) \subseteq \mathrm{Ker}\ f$, where $\pi:P \longrightarrow P/\mathrm{Ker}\ p$ is the natural epimorphism then $q(C_{p_{\{Q\}}}(\mathrm{Ker}\ p:P)) = Q \subseteq \mathrm{Ker}\ f$ - a contradiction since $f \neq 0$. Hence there is a homomorphism $u:Q \to P/\mathrm{Ker}\ p$ with $q(\pi^{-1}(\mathrm{Im}\ u)) \nsubseteq \mathrm{Ker}\ f$. Put $k = \overline{q} \circ u$, where $\overline{q}$ is induced by q and $g = \overline{p} \circ u$, where $\overline{p}$ is induced by p. Then $0 \neq h \circ g = \overline{f} \circ k$.

(vii) implies (ii). If there is a projective presentation $0 \to K \hookrightarrow P \to Q \to 0$ of Q and a submodule $X \subseteq P$ with P/X finitely cogenerated such that $K + C_{p_{\{Q\}}}(X:P) \neq P$ and $f:P/X \to P/(K + C_{p_{\{Q\}}}(X:P))$ is the natural epimorphism then there is a homomorphism $g:Q \to P/X$ with $\mathrm{Im}\ g \nsubseteq \mathrm{Ker}\ f$, a contradiction. Hence for every projective presentation $0 \to K \hookrightarrow P \to Q \to 0$ of Q and every submodule $X \subseteq P$ with P/X finitely cogenerated $K + C_{p_{\{Q\}}}(X:P) = P$.
The rest is either clear or follows from Propositions 1(i), 2, 3(iv) and 4.

Proposition 9. Let $Q \in R$-mod. Then the following are equivalent:

(i) Q is r dQF-3'',

(ii) there is a projective presentation $0 \to K \hookrightarrow P \to Q \to 0$ of Q such that $K + C_{\widetilde{p_{\{Q\}}}}(X:P) = P$ for every $X \subseteq P$ with P/X finitely cogenerated (coembedded),

(iii) whenever $A \subseteq B$, (B/A cocyclic) B finitely coembedded and $\mathrm{Hom}_R(Q,B) = 0$ then $\mathrm{Hom}_R(Q,B/A) = 0$.

Moreover, if Q has a projective cover then (i)-(iii) are equivalent to

(iv) $\mathrm{Hom}_R(Q,Y) \neq 0$ for every finitely coembedded nonzero factormodule Y of C(Q),

(v) $(Gh)(\widetilde{p_{\{Q\}}}) = p_{\{C(Q)\}}$,

(vi) $(Gh)(\widetilde{p_{\{Q\}}})$ is cohereditary,

(vii) $\widetilde{p_{\{Q\}}}(X) = p_{\{C(Q)\}}(X)$ for every finitely cogenerated (coembedded) module X,

(viii) $(Gh)(\widetilde{p_{\{Q\}}})(C(Q)) = C(Q)$,

(ix) for every finitely cogenerated (coembedded) module X $p_{\{C(Q)\}}(X) = X$ implies $Hom_R(Q,Y) \neq 0$ whenever Y is a nonzero factormodule of X,

(x) for every finitely coembedded module X $Hom_R(Q,X) = 0$ if and only if $Hom_R(C(Q),X) = 0$.

Proof. It can be led similarly as in Proposition 8.

Proposition 10. Let $Q \in R$-mod. If $p_{\{Q\}}$ has FCgSP then Q is dQF-3'' if and only if it is r dQF-3''.

Proof. It suffices to prove only the "only if" part. If Q is r dQF-3'' and there is a projective presentation $0 \to K \hookrightarrow P \to Q \to 0$ of Q, a submodule X of P with P/X finitely cogenerated and $K + C_{p_{\{Q\}}}(X:P) \neq P$ then $Hom_R(Q,P/(K + C_{p_{\{Q\}}}(X:P))) \neq 0$ and hence $Hom_R(Q,P/C_{p_{\{Q\}}}(X:P)) \neq 0$ by Proposition 9(iii). Thus there is a nonzero homomorphism $g:Q \to P/C_{p_{\{Q\}}}(X:P)$ which can be factorized through a homomorphism $h:Q \to P/X$, a contradiction. Thus Q is dQF-3'' by Proposition 8.

Proposition 11. Let S be a simple R-module possessing a projective cover. Then S is dQF-3'' if and only if it is projective.

Proof. Let $0 \neq S$ be a simple R-module with a projective cover $0 \to K \hookrightarrow P \to S \to 0$. If $X \subsetneqq P$ with P/X finitely cogenerated then $X \subseteq K$ since K is a maximal submodule of P and K is small in P. Further $p_{\{S\}}(P/X) = P/X$ by Proposition 8. Hence there is a homomorphism $f:S \longrightarrow P/X$ such that $\mathrm{Im}\, f \nsubseteq K/X$. Thus $\mathrm{Im}\, f = P/X$ and hence f is an isomorphism. Therefore $X = K$. Hence $K = 0$ and consequently S is projective. The converse is clear.

A module Q is called strongly dQF-3'' (strongly r dQF-3'') if there is a projective module P such that $(Gh)(p_{\{Q\}}) = p_{\{P\}}$ $((Gh)(\widetilde{p_{\{Q\}}}) = p_{\{P\}})$.

Proposition 12.

(i) Every strongly dQF-3'' (strongly r dQF-3'') module is dQF-3'' (r dQF-3'').

(ii) If a module Q has a projective cover then Q is strongly dQF-3'' (strongly r dQF-3'') if and only if it is dQF-3''(r dQF-3'').

(iii) A module Q is strongly dQF-3'' (strongly r dQF-3'') if and only if there is a projective representation $0 \longrightarrow K \hookrightarrow P \longrightarrow Q \longrightarrow 0$ of Q such that $(Gh)(p_{\{Q\}}) = p_{\{P\}}$ $((Gh)(\widetilde{p_{\{Q\}}}) = p_{\{P\}})$.

Proof. Obvious.

A module Q is said to be a G-generator if $p_{\{Q\}}(N) = N$ for every finitely cogenerated (coembedded) module N.

Remark 13. Let $Q \in R$-mod. Then Q is a G-generator if and only if $(Gh)(p_{\{Q\}}) = \mathrm{id}$.

Proposition 14. Let $Q \in R$-mod. Then the following are equivalent:

(i) Q is a G-generator,

(ii) Q is strongly dQF-3'' and every simple R-module is isomorphic to a factormodule of Q,

(iii) Q is dQF-3'' and every simple R-module is isomorphic to a factormodule of Q.
Moreover, if Q has a projective cover $(C(Q), \varphi_Q)$ then (i)-(iii) are equivalent to

(iv) Q is dQF-3'' and C(Q) is a generator.

Proof. (iii) implies (i). Suppose there is a finitely cogenerated module X with $p_{\{Q\}}(X) \neq X$. Then there is a cocyclic module C such that $0 \neq C \in \mathcal{F}_{p_{\{Q\}}}$ since $p_{\{Q\}}$ is G-cohereditary, a contradiction.
The rest is clear.

Remark 15. A projective module Q is a G-generator if and only if it is a generator.

Proposition 16. Let $Q = \sum^{\oplus}_{S \in \mathcal{S}} S$, where $\mathcal{S}$ is the representative set of simple left R-modules. Then the following are equivalent:

(i) Q is dQF-3'',

(ii) Soc is G-cohereditary.

(iii) Q is a G-generator,

(iv) R is a left V-ring.

Proof. It follows immediately from Proposition 14 and the fact that Soc = $p_{\{Q\}}$.

Let us Y denote a preradical defined by $Y(M) = \bigcap N$, where N runs through all submodules of M with M/N cocyclic and small in E(M/N).

Proposition 17. Y is a G-hereditary radical.

Proof. Obvious.

Proposition 18. Let Q be a cofaithful dQF-3'' with Y(Q) = = Q. Then $(Gh)(p_{\{Q\}}) = Y$.

Proof. $Y(Q) = Q$ implies $p_{\{Q\}} \leq Y$ and hence $(Gh)(p_{\{Q\}}) \leq Y$ by Proposition 17.
On the other hand if $r(N) = 0$, where $r = p_{\{Q\}}$, N finitely coembedded and $Y(N) \neq 0$ then there is a cocyclic factormodule C of N with $Y(C) \neq 0$. Thus C is not small in E(C) and hence there is a proper submodule K of E(C) with $C + K = E(C)$. Now r is G-cohereditary, $r(N) = 0$, N finitely coembedded. Hence $r(E(C)/K) = 0$ by Proposition 3(iv) since E(C)/K is isomorphic to a factormodule of N. Further Q is cofaithful and hence $E(C) \in \mathcal{T}_r$ and consequently $r(E(C)/K) = E(C)/K$, a contradiction. Thus $Y(N) = 0$. Therefore $Y(N) \subseteq r(N)$ for every finitely coembedded module N and hence $Y \leq (Gh)(p_{\{Q\}})$.

Proposition 19. Let R be a left perfect ring and Q be a cofaithful module. Then the following are equivalent:

(i) $(Gh)(p_{\{Q\}}) = Y$,

(ii) Q is dQF-3$''$ and $Y(Q) = Q$,

(iii) $\mathcal{T}_{(Gh)(p_{\{Q\}})} = \mathcal{T}_Y$.

Proof. (iii) implies (ii). $Y(Q) = Q$ by (iii). If $X \subseteq C(Q)$ such that $C(Q)/X$ is finitely cogenerated then $Y(C(Q)/X) =$ $= C(Q)/X$ since Y is cohereditary for a left perfect ring and hence $p_{\{Q\}}(C(Q)/X) = C(Q)/X$.

(ii) implies (i). By Proposition 18.

The rest is clear.

Proposition 20. Every direct sum of (strongly) dQF-3$''$ modules is (strongly) dQF-3$''$.

Proof. Obvious.

Proposition 21. Let $A, B \in$ R-mod. If $p_{\{A\}}(B) = B$ then the following are equivalent:

(i) $A \oplus B$ is dQF-3'',

(ii) A is dQF-3''.

Proof. Obvious.

Proposition 22. Let $Q \in$ R-mod. If every cocyclic factormodule of Q is dQF-3'' then Q is dQF-3''.

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