

Sergio Salbany

The completion monad and its algebra

Commentationes Mathematicae Universitatis Carolinae, Vol. 23 (1982), No. 2, 301--311

Persistent URL: <http://dml.cz/dmlcz/106152>

Terms of use:

© Charles University in Prague, Faculty of Mathematics and Physics, 1982

Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.



This paper has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* <http://project.dml.cz>

THE COMPLETION MONAD AND ITS ALGEBRA
Sergio SALBANY

Abstract: Let C represent the completion functor discussed by O. Wyler and S. Salbany. There is a monad associated with C and it is natural to ask for a characterization of the C -algebras. In this paper we show that the C -algebras are the complete spaces.

Key words: Quasi Uniform spaces, completion triple C , C -algebras.

Classification: Primary 54E15
Secondary 18C15

Introduction. As shown in [4] and [5], the completion functor on the category of Quasi-Uniform spaces is associated with a monad (C, η, μ) . Keith Hardie asked us for a characterization of the C -algebras and persuaded us, over the years, that an answer should be given.

We shall follow the terminology of [3] and [1] concerning quasi-uniform spaces and that of [2] for the category theory.

1. Definitions, constructions and notations

A. Cauchy filters, convergence and completeness. Let QU denote the category of quasi-uniform spaces (X, \mathcal{U}) and quasi-uniformly continuous maps.

Definition 1. A filter \mathcal{F} on (X, \mathcal{U}) is said to be a

Cauchy-filter if, for every U in the uniformity $\mathcal{U} \vee \mathcal{U}^{-1}$ there is F in \mathcal{F} such that $F \times F \subset U$.

Definition 2. A filter \mathcal{F} on (X, \mathcal{U}) is said to converge to x if it converges to x in the topology induced by the uniformity $\mathcal{U} \vee \mathcal{U}^{-1}$.

Definition 3. (X, \mathcal{U}) is said to be complete if $\mathcal{U} \vee \mathcal{U}^{-1}$ is complete.

Definition 4. If A is a subset of (X, \mathcal{U}) , denote by \bar{A} the closure of A in the topology induced by the uniformity $\mathcal{U} \vee \mathcal{U}^{-1}$. x is called an adherence point of a filter \mathcal{F} if $x \in \bar{F}$ for every F in \mathcal{F} .

Note. As for uniform spaces, if \mathcal{F} is a Cauchy filter on (X, \mathcal{U}) and x is an adherence point of \mathcal{F} , then \mathcal{F} converges to x , and conversely. Thus, if $x \in \bar{F}$ for all $F \in \mathcal{F}$, then \mathcal{F} converges to x .

B. Description of the completion monad. Given (X, \mathcal{U}) , let CX denote the set of all Cauchy filters on X and let \mathcal{U}^* denote the quasi-uniformity on CX with basis elements U^* , where $U \in \mathcal{U}$ and $(\alpha, \beta) \in U^*$ if and only if there are sets A in α , B in β such that $A \times B \subset U$. The sets U^* do form a basis for \mathcal{U}^* since $(U \cap V)^* = U^* \cap V^*$. We now describe the multiplication μ and the unity η of the triple C :

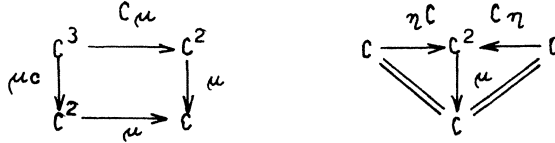
(i) Let $\eta_x: X \rightarrow CX$ be given by $\eta_x(x) = \{F \mid F \subset X \text{ and } x \in F\}$. Then $\eta_x: (X, \mathcal{U}) \rightarrow (CX, \mathcal{U}^*)$ is quasi-uniformly continuous.

(ii) Let $\mu_x: C^2X \rightarrow CX$ be given by $\mu_x(\alpha) = \{H \mid H \subset X, H^* \in \alpha\}$, where H^* is such that $\beta \in H^*$ if and only if $H \in \beta$. Then $\mu_x: (C^2X, \mathcal{U}^{**}) \rightarrow (CX, \mathcal{U}^*)$ is quasi-uniformly continuous.

Moreover,

(a) η_X is an initial and injective map onto a $\mathcal{U}^* \vee (\mathcal{U}^*)^{-1}$ -dense subspace of CX .

(b) μ_X and η_X induce natural transformations $\mu: C^2 \rightarrow C$; $\eta: \mathbb{1} \rightarrow C$ such that the following diagrams commute



Thus, every space can be densely embedded in a complete space in a "regular" way, which is expressed by the functoriality of C . Moreover, even though the completion process always enlarges a space, the existence of μ shows that the completion of a complete space is not "much larger" than the complete space itself.

C. The separated completion

Definition 5. A quasi-uniform space (X, \mathcal{U}) is separated if the uniformity $\mathcal{U} \vee \mathcal{U}^{-1}$ is separated, that is, the intersection of all members of $\mathcal{U} \vee \mathcal{U}^{-1}$ is the diagonal of $X \times X$.

Construction of the separated reflection. Given a quasi-uniform space (X, \mathcal{U}) , let R denote the equivalence relation $x R y$ if and only if $\{\bar{x}\} = \{\bar{y}\}$. Denote by $[x]$, the R -equivalence class of x . Let X^s denote the set of R -equivalence classes on (X, \mathcal{U}) . Let $s: X \rightarrow X^s$ denote the map $s(x) = [x]$. For $U \in \mathcal{U}$, let $U^s = \{([x], [y]) \mid (x, y) \in U\}$, then the U^s form a basis for a quasi-uniformity \mathcal{U}^s (since $(U \cap V)^s \subset U^s \cap V^s$). The map $s_x: (X, \mathcal{U}) \rightarrow (X^s, \mathcal{U}^s)$ is an initial quasi-uniformly continuous map onto a separated quasi-uniform space

and the assignment $(X, \mathcal{U}) \longrightarrow (X^s, \mathcal{U}^s)$ is the separated reflection in QU. Moreover, (X, \mathcal{U}) is complete if and only if (X^s, \mathcal{U}^s) is complete. The composite $s \circ C = C^s$ is the separated-completion-functor. The natural transformations $\eta^s = s\eta$ and $\mu^s = s\mu$ provides the separated-completion monad (C^s, η^s, μ^s) .

The importance of separated completions lies in the fact that the embedding map $\eta_X^s: (X, \mathcal{U}) \longrightarrow (X^s, \mathcal{U}^s)$ is a map onto a $\mathcal{U}^s \vee (\mathcal{U}^s)^{-1}$ -dense subspace, hence an epimorphism in the category of separated quasi-uniform spaces (see [1],[3]).

D. The algebra of a monad

Definition 6. Let (C, η, μ) be a monad on a category A. An object A of A is a C -algebra if there is a morphism h in A, called a structure map, $h: CA \longrightarrow A$ such that the following diagrams commute:

$$\begin{array}{ccc}
 C A & \xrightarrow{C h} & C A \\
 \mu_A \downarrow & & \downarrow h \\
 C A & \xrightarrow{h} & A
 \end{array}
 \qquad
 \begin{array}{ccc}
 A & \xrightarrow{\eta_A} & C A \\
 \parallel & & \downarrow h \\
 & & A
 \end{array}$$

Examples of C-algebras

Example 1. Let $X = \{0, 1\}$ and $\mathcal{U} = X \times X$. It is straightforward to verify that (X, \mathcal{U}) is a C -algebra for every map $h: (CX, \mathcal{U}^*) \longrightarrow (X, \mathcal{U})$.

Example 2. Let (X, \mathcal{U}) be a separated complete space and let $h: (CX, \mathcal{U}^*) \longrightarrow (X, \mathcal{U})$ be the limit map, $h(\mathcal{F}) = \text{limit of } \mathcal{F}$ (convergence in the topology of $\mathcal{U} \vee \mathcal{U}^{-1}$). Note that h is

well defined since limits are unique in separated quasi-uniform spaces. It will be shown in Section 3 that such a separated complete space is a C-algebra.

2. C-algebras. As expected, C-algebras are the complete spaces. Although expected, we have found the proof elusive. The result is surprising in that there is an arbitrariness in the structure map $h: CX \rightarrow X$ (see Proposition 2) that suggests that not all complete spaces would be C-algebras.

Proposition 1. If (X, \mathcal{U}) is a C-algebra, then (X, \mathcal{U}) is complete.

This is an immediate consequence of the following two lemmas.

Lemma 1. If $S \subset X$, then $S^* \subset \overline{\eta_X[S]}$.

Proof. Let $\alpha \in S^*$. Given a symmetric U in $\mathcal{U} \vee \mathcal{U}^{-1}$, we show that $U[\alpha]$ intersects $\eta_X[S]$. From $\alpha \in S^*$ it follows that $S \in \alpha$. Now α is a Cauchy-filter, so there is F in α such that $F \times F \subset U$. But S and F are in α so there is x in $F \cap S$. By definition of U^* it follows that $(\alpha, \eta_X(x)) \in U^*$ since $F \in \alpha$ and $F \in \eta_X(x)$ and $F \times F \subset U$. Thus $U^*[\alpha]$ intersects $\eta_X[S]$, as required.

Lemma 2. Let $h: (CX, \mathcal{U}^*) \rightarrow (X, \mathcal{U})$ be a quasi-uniformly continuous map such that $h \circ \eta_X(x) = x$ for all x in X . Then a Cauchy filter \mathcal{F} converges to $h(\mathcal{F})$.

Proof. We show that $h(\mathcal{F})$ is an adherence point of \mathcal{F} , from which it follows that \mathcal{F} converges to $h(\mathcal{F})$. Let $S \in \mathcal{F}$, then $F \in S^*$, so that $h(\mathcal{F}) \in h[S^*]$. By Lemma 1 we have $h[S^*] \subset \overline{h[\eta_X[S]]} \subset \overline{h[\eta_X[S]]} = \overline{S}$. Thus $h(\mathcal{F}) \in \overline{S}$ for all S in \mathcal{F} ,

showing that $h(\mathcal{F})$ is an adherence point of \mathcal{F} .

To establish the converse of Proposition 1 we require the following lemmas.

Lemma 3. Let $\alpha \in C^2X$, then $\bigcap \{ \overline{H} \mid H \subset X \text{ and } h \leftarrow [H] \in \alpha \} = \bigcap \{ \overline{h[k]} \mid k \in \alpha \}$.

Proof. Suppose $H \subset X$ and $h \leftarrow [H] \in \alpha$. $h[h \leftarrow [H]] = H$ (since h is surjective) shows that $\overline{H} = \overline{h[k]}$ for some k in α , since $k \in \alpha$ and $K \subset h \leftarrow [h[k]]$. The proof is complete.

Lemma 4. Let $\alpha \in C^2X$. If α converges to \mathcal{F} , then $ch(\alpha)$ converges to $h(\mathcal{F})$.

Proof. Suppose α converges to \mathcal{F} , then \mathcal{F} is an adherence point of α so that $\mathcal{F} \in \overline{K}$ for all $k \in \alpha$. Hence $h(\mathcal{F}) \in h[\overline{K}] \subset \overline{h[k]}$ for all $k \in \alpha$. Thus, by Lemma 3, $h(\mathcal{F}) \in \bigcap \{ \overline{H} \mid h \leftarrow [H] \in \alpha \}$, so that $h(\mathcal{F})$ is an adherence point of $Ch(\alpha)$, as required.

Lemma 5. Let $V \in \mathcal{U} \vee \mathcal{U}^{-1}$. Let U be symmetric and such that $U \circ U \subset V$. If \mathcal{F} is a Cauchy filter which converges to x , then $U^*[\mathcal{F}] \subset (V[x])^*$.

Proof. Because \mathcal{F} is Cauchy and converges to x , there is F in \mathcal{F} such that $F_1 \times F_1 \subset U$ and $F_1 \subset U[x]$. Suppose $\chi \in U^*[\mathcal{F}]$, we show that $V[x] \in \chi$. By definition of U^* , if $\chi \in U^*[\mathcal{F}]$, there is $F_2 \in \mathcal{F}$ and $G \in \mathcal{F}$ such that $F_2 \times G \subset U$. Let $F = F_1 \cap F_2$ so that (i) $F \in \mathcal{F}$, (ii) $F \times G \subset U$, and (iii) $F \subset U[x]$.

From (ii) we have $G \subset U[F]$ so that $U[F] \in \chi$ since $G \in \chi$. But $U[F] \subset U \circ U[x] \subset V[x]$, from (iii), so that $V[x] \in \chi$,

as required.

Lemma 6. Let $\alpha \in C^2X$. If α converges to \mathcal{F} and \mathcal{F} converges to x , then $\mu_x(\alpha)$ converges to x .

Proof. Let $V \in \mathcal{U} \vee \mathcal{U}^{-1}$. We show that $V[x] \in \mu_x(\alpha)$. Choose a symmetric entourage U such that $U \circ U \subset V$. Because \mathcal{F} converges to x , by Lemma 5, we have $U^*[\mathcal{F}] \subset (V[x])^*$. Since α converges to \mathcal{F} we also have $U^*[\mathcal{F}]$ in α , hence $(V[x])^*$ is in α and, consequently, $V[x] \in \mu_x(\alpha)$, as required.

Lemma 7. Let $\alpha \in C^2X$, if there is $x \in X$ such that $\mu_x(\alpha) = \{H \mid x \in H\}$, then $\text{Ch}(\alpha) = \mu_x(\alpha)$.

Proof. Observe that, for any filter \mathcal{F} , $\mathcal{F} = \{H \mid x \in H\} \iff \{x\} \in \mathcal{F}$. For convenience, let $\{H \mid x \in H\}$ be denoted by $\langle x \rangle$. To show that $\text{Ch}(\alpha) = \langle x \rangle$ it suffices to prove that $\{x\} \in \text{Ch}(\alpha)$, that is, $h \leftarrow [\{x\}] \in \alpha$.

Now, $\langle x \rangle \in \alpha$ since $\{x\} \in \mu(\alpha)$ implies $(\{x\})^* \in \alpha$ and $\mathcal{F} \in (\{x\})^*$ is equivalent to $\{x\} \in \mathcal{F}$ which states that $\mathcal{F} = \langle x \rangle$, so that $\langle x \rangle \in \alpha$. Now $\langle x \rangle \in h \leftarrow [\{x\}]$ since $h(\langle x \rangle) = x$. It then follows that $h \leftarrow [\{x\}] \in \alpha$, as required.

Note: Consideration of example 1 shows that if $\text{Ch}(\alpha) = \{H \mid x \in H\}$, then it does not follow that $\mu_x(\alpha) = \text{Ch}(\alpha)$.

Lemma 8. Let A_x consist of all Cauchy filters on X which converge to x . Let $\alpha \in C^2X$. If $A_x \in \alpha$, then $\mu_x(\alpha)$ converges to x .

Proof. It is straightforward to check $A_x = \bar{A}_x$. Also CX is complete so there is \mathcal{F} in CX such that α converges to \mathcal{F} . Then \mathcal{F} is an adherence point to α , so that $\mathcal{F} \in \bar{A}_x$. But

then $\mathcal{F} \in \mathcal{A}_x$, so that \mathcal{F} converges to x . By Lemma 6, we have that $\mu_x(\alpha)$ converges to x .

Proposition 2. If (X, \mathcal{U}) is complete, then it is a C-algebra.

Proof. Define $h: CX \rightarrow X$ as follows: For each x , let $[x]$ denote the R equivalence class of x ($xRy \iff \bar{x} = \bar{y}$). Let c be a choice function on $X^{\mathcal{A}}$, so $c([x]) \in [x]$. Observe that if \mathcal{F} is a Cauchy filter on X which converges to x and y , then $[x] = [y]$; now let

$$h(\mathcal{F}) = \begin{cases} x, & \text{if } \mathcal{F} = \eta_x(x) \text{ for some } x. \\ c([x]), & \text{if } \mathcal{F} \notin \eta_x[X] \text{ and } \mathcal{F} \text{ converges to } x. \end{cases}$$

By the remarks above, h is well defined. Observe that \mathcal{F} converges to $h[\mathcal{F}]$. It is readily checked that $h: (CX, \mathcal{U}^*) \rightarrow (X, \mathcal{U})$ is quasi-uniformly continuous. We verify that the diagram corresponding to (1) in the definition of the algebra of a monad is commutative: Let $\alpha \in \mathcal{C}^2X$. Because CX is complete, α converges to some \mathcal{F} in CX .

By Lemma 4, $Ch(\alpha)$ converges to $h(\mathcal{F})$. By Lemma 6, $\mu_x(\alpha)$ converges to $h(\mathcal{F})$ and α converges to \mathcal{F} . Thus $\mu_x(\alpha)$ and $Ch(\alpha)$ both converge to $h(\mathcal{F})$. To show that $h[\mu_x(\alpha)] = h[Ch(\alpha)]$ we consider two cases.

Case 1. $Ch(\alpha) \notin \eta_x[X]$. Then $\mu_x(\alpha) \notin \eta_x[X]$, by Lemma 7. By definition of h , it follows that $h[\mu_x(\alpha)] = c([h(\mathcal{F})])$ and $h[Ch(\alpha)] = c([h(\mathcal{F})])$, as required.

Case 2. $Ch(\alpha) \in \eta_x[X]$. Then $Ch(\alpha) = \langle x \rangle$ (for a unique x in X) so that $h \leftarrow [\{x\}] \in \alpha$. We again distinguish

two cases:

Case 2 a. $x \neq c([x])$. In this case, $h \leftarrow [\{x\}] = \{ \langle x \rangle \}$ since for any \mathcal{F} not in $\eta_x[X]$, $h(\mathcal{F}) = x$ implies \mathcal{F} converges to x so that $h(\mathcal{F}) = c([x]) \neq x$. Now $Ch(\alpha) = \langle x \rangle$ implies that $h \leftarrow [\{x\}] \in \alpha$, so that $\{ \langle x \rangle \} \in \alpha$. But then $\mu_x(\alpha) = \langle x \rangle$. Thus $Ch(\alpha) = \mu_x(\alpha) = \langle x \rangle$ so that $h(Ch(\alpha)) = h(\mu_x(\alpha))$.

Case 2 b. $x = c([x])$. In this case $h \leftarrow [\{x\}]$ consists of all Cauchy filters \mathcal{F} which converge to a point in $[x]$. Thus $h \leftarrow [\{x\}] = A_x$. From Lemma 8, it follows that $\mu_x(\alpha)$ converges to x . From the definition of h (whether or not $\mu_x(\alpha)$ is in $\eta_x[X]$) and $x = c([x])$, it follows that $h[\mu_x(\alpha)] = x$. Hence $h[\mu_x(\alpha)] = h[Ch(\alpha)]$.

It is remarkable that the regularity expressed in diagram 1 of the definition of a monad could be achieved with the arbitrariness involved in the function h of Proposition 2.

3. The separated-completion C^s . The results in Section 2 readily identify the C^s -algebras in QU_s :

Proposition 3. The C^s -algebras are the separated and complete spaces.

However, the delicate comparison of filters and limits can be avoided and a simple proof of Proposition 3 will be given in two parts, where we rely on the fact that η_x is epic in QU_s .

Part 1. Every C^s -algebra is complete and separated.

Proof. Let $h: C^s X \rightarrow X$ be the structure map. Then $h \circ \eta_x = \eta_x$, so that $\eta_x \circ h \circ \eta_x = \eta_x \circ \eta_x = \eta_x \circ \eta_x$. But η_x is an

epimorphism, hence $\eta_x \circ h = \mathbb{1}_{C^S X}$. Thus X and $C^S X$ are isomorphic.

Part 2. Let X be a complete and separated space, then every Cauchy filter \mathcal{F} converges to a unique point x . It is straightforward to verify that $h: (C^S X, (\mathcal{U}^*)^S) \rightarrow (X, \mathcal{U})$ is quasi-uniformly continuous. Moreover $h \circ \eta_x = \mathbb{1}_x$, since the filter $\eta_x(x)$ converges to x .

To prove that $h \circ \mu_x = h \circ Ch$ we show that $h \circ \mu_x \circ \eta_{Cx} = h \circ Ch \circ \eta_{Cx} (=h)$ and use the fact that η_{Cx} is epic.

Now $\mu_x \circ \eta_{Cx} = \mathbb{1}_{Cx}$, so that $h \circ \mu_x \circ \eta_{Cx} = h$. Also, $Ch \circ \eta_{Cx} = \eta_x \circ h$ (by naturality of η), so that

$$h \circ Ch \circ \eta_{Cx} = h \circ \eta_x \circ h = \mathbb{1}_x \circ h = h.$$

The proof is complete.

R e f e r e n c e s

- [1] G.C.L. BRUMMER: A categorical study of initiality in uniform topology, Thesis, Univ. of Cape Town, 1971.
- [2] S. MACLANE: Categories for the working mathematician, Springer-Verlag, New York, Berlin, 1971.
- [3] S. SALBANY: Bitopological spaces, compactifications and completions, Math. Monographs U.C.T. Vol. 1, 1974.
- [4] S. SALBANY: Completions and triples, Math. Colloq. U.C.T. VII(1973), 55-61.
- [5] O. WYLER: Filter space monads, regularity, completion, TOPO 72 - General Topology and its Applications, Lecture Notes in Math. 378(1974), 591-637.

University of Zimbabwe,
Salisbury,
Zimbabwe

(Oblatum 9.11. 1981)