Pavel Drábek
Solvability of nonlinear problems at resonance

Commentationes Mathematicae Universitatis Carolinae, Vol. 23 (1982), No. 2, 359--368

Persistent URL: http://dml.cz/dmlcz/106159

Terms of use:
© Charles University in Prague, Faculty of Mathematics and Physics, 1982

Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these Terms of use.
Abstract: This paper deals with the solvability of nonlinear operator equations with finite-dimensional kernel of the linear part and with nonlinearity given by odd real function $g$ with $\int_0^\infty g(z)dz \in \mathbb{R} \cup \{ \pm \infty \}$ and with no restrictions on $\lim_{t \to \infty, \tau (\omega, t)} t \min g(\tau)$.

Key words: Noncoercive problems at resonance, weakly nonlinear boundary value problems, vanishing nonlinearities

Classification: 47H15, 35J40

-----------------------------

1. Assumptions. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, $H = L^2(\Omega)$ be the real Hilbert space with usual inner product $\langle \cdot, \cdot \rangle$ and with the norm $\|u\| = \langle u, u \rangle^{1/2}$. Suppose that

$L: D(L) \subset H \rightarrow H$

is a symmetric linear operator with dense domain $D(L)$, with nontrivial finitedimensional nullspace $N(L)$ and closed range $R(L)$. Let

$H = N(L) \oplus R(L)$

and suppose that

$K = (L|R(L))^{-1}: R(L) \rightarrow R(L)$

(so called the right inverse of $L$) is completely continuous.

We assume that $N(L)$ has "unique continuation property" in the sense that the only function $w \in N(L)$ vanishing on $a$
Let $G$ be the Nemytskii operator associated with continuously differentiable odd bounded function $g: \mathbb{R} \to \mathbb{R}$, $g \neq 0$,
\[ G: u \mapsto g \circ u. \]
Obviously $G$ maps $H$ into $H$ and has bounded range.

Let us suppose that
(1) $c = \|K\| \sup_{x \in \mathbb{R}} |g'(x)| < 1$,
(2) there exists $\int_{0}^{+\infty} g(z)dz$.

Let us denote $I = \int_{0}^{+\infty} g(z)dz$ (we admit $I = \pm \infty$).

In distinction from papers [13] and [12] we assume nothing about the limit
\[ \lim_{t \to \infty} \min_{\alpha, \beta} g(\tau). \]

This paper also generalizes in some sense the results from [3], [4] and [6] because we may have $\dim N(L) > 1$.

2. **Theorem.** Let $f \in R(L)$. Then the operator equation
(3) $Lu + G(u) = f$
has at least one solution.

3. **Proof of the theorem.** We use the global Lyapunov-Schmidt method. For this purpose we denote $P$ and $Q$ the orthogonal projections from $H$ onto $N(L)$ and $R(L)$, respectively. It is easy to see that the solvability of (3) is equivalent to the solvability of the bifurcation system
(3a) $v + KQG(w + v) - Kf = 0$,
(3b) $PG(w + v) = 0$, 
- 360 -
\( w \in N(L), v \in R(L), w = Pu, v = Qu. \)

**Step 1.** For each \( w \in N(L) \) there exists exactly one \( v(w) \in R(L) \) such that

\[
(3a) \quad v(w) + KQG(w + v(w)) - Kr = 0.
\]

Define \( F(w,.) : R(L) \rightarrow R(L), \)

\[
F(w,v) : v \mapsto Kr - KQG(w + v),
\]

for each \( w \in N(L) \). Then using Hölder inequality we obtain that

\[
\|F(w,v_1) - F(w,v_2)\| \leq \|K\|\|Q\|\sup_{u \in R} \int |g(w + v_1) - g(w + v_2)| |u| \leq \\
\leq \|K\|\sup_{u \in R} \int |g(z)| |v_1 - v_2| = c \|v_1 - v_2\|
\]

holds for each \( w \in N(L), v_1, v_2 \in R(L) \). The Banach contraction theorem implies that for each \( w \in N(L) \) there exists exactly one \( v(w) \in R(L) \) that

\[
v(w) = F(w,v(w)).
\]

**Step 2.** There exists \( r > 0 \) such that for each \( w \in N(L) \) it is

\[
(4) \quad \|v(w)\| \leq r.
\]

The proof follows immediately from the boundedness of \( G \).

**Step 3.** It is

\[
(5) \quad \lim_{\ell \to +\infty} \text{meas} \{ x \in \Omega ; |v(w)(x)| \geq \ell \} = 0,
\]

uniformly with respect to \( w \in N(L) \).

The equality (5) follows from (4).

**Step 4.** For each \( k \in \mathbb{N} \) we have

\[
\lim_{\|w\| \to \infty} \text{meas} \{ x \in \Omega ; |w(x)| \leq k \} = 0.
\]

- 361 -
Suppose on the contrary that there exists $k_0 \in \mathbb{N}$, $w_n \in N(L)$, 

$$\|w_n\| \to +\infty$$

such that

$$\text{meas} \left\{ x \in \Omega ; |w_n(x)| \leq k_0 \right\} \geq \varepsilon_0 > 0.$$  

Put $\hat{w}_n = w_n / \|w_n\|$. Then we have

$$\text{meas} \left\{ x \in \Omega ; |\hat{\hat{w}}_n(x)| \leq k_0 / \|w_n\| \right\} \geq \varepsilon_0.$$  

Since $\dim N(L) < +\infty$ we can suppose that $\hat{w}_n \to w_0$ in $L^2(\Omega)$, i.e. by Jégorov's theorem for each $\eta > 0$ there exists $\Omega' \subset \Omega$, $\text{meas} \Omega' \leq \eta$ and $\hat{w}_n \to w_0$ (uniformly) on $\Omega \setminus \Omega'$. If we put $\eta = \varepsilon_0 / 2$ and take the limit for $n \to +\infty$ in (6), we obtain

$$\text{meas} \left\{ x \in \Omega ; |w_0(x)| = 0 \right\} \geq \varepsilon_0 / 2 > 0,$$

which is a contradiction with $w_0 \in N(L)$ and the unique continuation property of $N(L)$.

**Step 5.** If $I \subset \mathbb{R}$ then it is

$$\lim_{\|w\| \to +\infty} v(w) = K_I \quad \text{and} \quad \lim_{\|w\| \to +\infty} L v(w) = g.$$

Using Hölder inequality we obtain

$$\|v(w) - K_I\|^2 \leq \|K\|^2 \left( \sup_{\|w\| \leq 1} \int_\Omega |g(w + v(w))| \right)^2 \leq \|K\|^2 \left( \int_\Omega |g(w + v(w))| \right)^2;$$

analogously $\|L v(w) - f\|^2 \leq \left( \int_\Omega |g(w + v(w))| \right)^2$.

Choose $\varepsilon > 0$. Then there exists $k > 0$ such that

$$\left( \sup_{|z| \leq \varepsilon} |g(z)| \right)^2 \text{meas } \Omega < \varepsilon / 2.$$  

According to **Steps 3 and 4** we obtain the existence of such $\varepsilon > 0$ that for $\|w\| \geq \varepsilon$ it is

$$\text{meas } \Omega_k = \text{meas} \left\{ x \in \Omega ; |w(x) + v(w)(x)| \leq k \right\} < \varepsilon / (2 \sup_{|z| \leq \varepsilon} |g(z)|^2).$$
Using (7) and (8) we obtain

\[ \| v(w) - Kf \|^2 \leq \| K \|^2 \left\{ \int_{\Omega \setminus \Omega_k} |g(w + v(w))|^2 \right\} + \\
+ \left( \int_{\Omega \setminus \Omega_k} |g(w + v(w))|^2 \right) + \left( \sup_{z \in \mathbb{R}} |g(z)|^2 \text{ meas } \Omega_k \right) + \\
+ \left( \sup_{z \in \mathbb{R}} |g(z)|^2 \text{ meas } \Omega \right) < \| K \|^2 \varepsilon ; \]

analogously we obtain \( \| Lv(w) - f \|^2 < \varepsilon \).

**Step 6.** Put

\[ \varphi(w) = \frac{1}{2} \langle Lv(w), v(w) \rangle + \int_{\Omega} \int_{0} |w + v(w)| g(z) dz - \int_{\Omega} f v(w). \]

Then

\[ \lim_{\| w \| \to \infty} \varphi(w) = \text{Im} \| \Omega - \frac{1}{2} \langle f, Kf \rangle, \text{ in the case } I \in \mathbb{R} \text{ and } \lim_{\| w \| \to \infty} \varphi(w) = \pm \infty, \text{ if } I = \pm \infty. \]

We shall prove the assertion for \( I \in \mathbb{R} \) and \( I = + \infty \) (the case \( I = - \infty \) is analogous). Let \( I \in \mathbb{R} \). According to Step 5 it is

\[ \lim_{\| w \| \to \infty} \left[ \frac{1}{2} \langle Lv(w), v(w) \rangle - \int_{\Omega} f v(w) \right] = - \frac{1}{2} \langle f, Kf \rangle. \]

Choose \( \varepsilon > 0 \). There exists \( k > 0 \) such that

\[ \left| \int_{0}^{+\infty} g(z) dz - I \right| < \varepsilon. \]

Let \( \sigma > 0 \) be such that (see Steps 3, 4)

\[ \text{meas } \Omega_k < \varepsilon, \]

for all \( w \in N(L) \), \( \| w \| < \sigma \). Then for \( \| w \| \geq \sigma \) we obtain using (9) and (10)

\[ \int_{\Omega} \int_{0} |w + v(w)| g(z) dz - \text{Im } \| \Omega \| \leq \int_{\Omega} \int_{0} |w + v(w)| g(z) dz - \\
- \text{Im } \| \Omega \| + \int_{\Omega} \int_{0} g(z) dz + \int_{\Omega} g(z) dz + \text{Im } \| \Omega \| < \\
\varepsilon \left( \text{meas } \Omega + \int_{0}^{\sigma} |g(z)| dz + I \right), \text{ which implies } \\
\lim_{\| w \| \to \infty} \int_{\Omega} \int_{0} |w + v(w)| g(z) dz = \text{Im } \| \Omega \|. \]
Let $I \to +\infty$. Then for arbitrary $\ell > 0$ there exists $k > 0$ such that
\[ \int_0^{+\infty} g(z)dz > \ell. \]

Let $\varepsilon > 0$ be such that $\text{meas } \Omega_k < \min \left(\frac{1}{\ell} \int_0^{+\infty} |g(z)|dz, \frac{1}{2} \text{meas } \Omega \right)$, for all $w \in N(L), \|w\| \geq \varepsilon$. Thus for $\|w\| \geq \varepsilon$ it is
\[ \int_0^a dx \int_0^{+\infty} g(z)dz \leq \int_0^a dx \int_0^{+\infty} g(z)dz - \left| \int_0^{+\infty} dx \int_0^{+\infty} g(z)dz \right| \geq \ell \text{meas } (\Omega \setminus \Omega_k) - \text{meas } \Omega_k \int_0^{+\infty} |g(z)|dz \geq 1/2 \ell \text{meas } \Omega - 1/\ell, \]
which implies
\[ \lim_{\|w\| \to \infty} \int_0^a dx \int_0^{+\infty} g(z)dz = +\infty. \]

This together with Step 2 proves the assertion for $I = +\infty$.

**Step 7.** The function $v(\cdot): w \mapsto v(w)$ is Fréchet differentiable on $N(L)$. Since $c < 1$ (see (1)), the Fréchet derivative of
\[ (v,w) \mapsto v - F(v,w) \]
with respect to the first variable is invertible (lemma of Minty) and the assertion then follows from the implicit function theorem.

According to Step 6 the function $\varphi: N(L) \to \mathbb{R}$ must attain its maximum or minimum in some point $w_0 \in N(L)$, if $I \in \mathbb{R}$, $\varphi$ attains its maximum for $I = -\infty$ and minimum for $I = +\infty$. Then
\[ \langle \varphi'(w_0), h \rangle = 0 \]
for each $h \in N(L)$. On the other hand, it is
\[ \langle \varphi'(w_0), h \rangle = 1/2 \langle Lv'(w_0)h, v(w_0) \rangle + 1/2 \langle Lv(w_0), v'(w_0)h \rangle + \cdots - 364 - \]
+ \int_{\Omega} g(w_0 + v(w_0))h + \int_{\Omega} g(w_0 + v(w_0))v'(w_0)h - \int_{\Omega} fv'(w_0)h. 

Since \( L \) is symmetric, it is 

\[ \frac{1}{2} \langle Lv'(w_0)h, v(w_0) \rangle + \frac{1}{2} \langle Lv(w_0), v'(w_0)h \rangle = \langle Lv(w_0), v'(w_0)h \rangle \]

and (because of \( v'(w_0)h \in R(L) \) and (3a) holds) 

\[ \langle Lv(w_0), v'(w_0)h \rangle + \int_{\Omega} g(w_0 + v(w_0))v'(w_0)h = \int_{\Omega} fv'(w_0)h \]

for each \( h \in \mathbb{N}(L) \). From (11) we obtain that 

\[ \int_{\Omega} g(w_0 + v(w_0))h = 0, \]

for each \( h \in \mathbb{N}(L) \), which is nothing else than (3b).

The function \( u = w_0 + v(w_0) \) is then the solution of (3).

4. Applications. The results of this paper may be applied, for instance, to the following types of semilinear elliptic boundary value problems:

(12) \[
\begin{cases}
- \Delta u - \lambda_k u + \beta u e^{-u^2} = f \text{ in } \Omega, \\
u = 0 \text{ on } \partial \Omega;
\end{cases}
\]

(13) \[
\begin{cases}
- \Delta u - \lambda_k u + \beta e^{-u^2} \sin u = f \text{ in } \Omega, \\
u = 0 \text{ on } \partial \Omega;
\end{cases}
\]

(14) \[
\begin{cases}
- \Delta^2 u - \lambda_k u + \frac{\beta u}{1 + u^2} = f \text{ in } \Omega, \\
u = \frac{\partial u}{\partial n} = 0 \text{ on } \partial \Omega;
\end{cases}
\]

(15) \[
\begin{cases}
- \Delta^2 u - \lambda_k u + g(u) = f \text{ in } \Omega, \\
u = \frac{\partial u}{\partial n} = 0 \text{ on } \partial \Omega,
\end{cases}
\]

where \( g \) is bounded, odd, continuously differentiable function with compact support in \( \mathbb{R} \).

- 365 -
We put \( D(L) = W_0^{1,2}(\Omega) \), resp. \( D(L) = W_0^{2,2}(\Omega) \), in the cases (12), (13), resp. (14), (15). The operator \( L \) is defined by

\[
\langle Lu, v \rangle = \int_\Omega \nabla u \nabla v - \lambda_k \int_\Omega uv,
\]

in cases (12) and (13);

\[
\langle Lu, v \rangle = \int_\Omega \Delta u \Delta v - \lambda_k \int_\Omega uv,
\]

in the cases (14) and (15). We suppose that \( \lambda_k \) is any eigenvalue of the Laplace operator \( \Delta \), resp. the biharmonic operator \( \Delta^2 \), with Dirichlet boundary conditions. Then the operator \( L \) satisfies all the assumptions from Section 1. Let us note that the assumption of "unique continuation property" is satisfied according to the result of Sitnikova [7].

The constant \( \beta > 0 \) depends on \( \Omega \) and it must be such that the assumption (1) is fulfilled.

5. Remarks. As it was pointed out in Section 1, we assume nothing about the limit

\[
(16) \quad \lim_{t \to \infty} \inf_{\tau \in (\omega, t]} g(\tau).
\]

It means that this paper generalizes the results of Fučík, Krášec [1] and Hess [2]. The price we must pay for this generalization is the assumption (1) which is not very eligible.

This paper generalizes the results of de Figueiredo, Ni [13] and Concalves [6] because we may have \( \dim N(L) > 1 \) and it need not be necessarily \( g(t)t \geq 0 \), \( t \in \mathbb{R} \).

Following the proof of the theorem it is obvious that the assumption that \( g \) is odd can be replaced by the assumption

\[
\int_0^\infty g(z) dz = - \int_0^\infty \overline{g}(z) dz.
\]

Studying the function \( \varphi : N(L) \to \mathbb{R} \) and using the
Brouwer degree theory it is possible to prove the existence of multiple solutions of (3) with the right hand side

\[ f = f_1 + f_2, \]

\( f_1 \in \mathbb{R}(L) \) and \( f_2 \in \mathbb{N}(L) \) with sufficiently small \( \| f_2 \| \). The sketch of the proof is given in [5].

6. Open problem. According to the author's best knowledge it remains to be an open problem to prove the theorem without the condition (1) which makes restriction on the derivative \( |g'(z)|, z \in \mathbb{R} \).

References


Katedra matematiky VŠSE, Nejedlého sady 14, 30614 Plzeň, Československá

(Oblatum 4.2. 1982)