Roger Yue Chi Ming
On von Neumann regular rings. VII.

Commentationes Mathematicae Universitatis Carolinae, Vol. 23 (1982), No. 3, 427--442

Persistent URL: http://dml.cz/dmlcz/106165

Terms of use:
© Charles University in Prague, Faculty of Mathematics and Physics, 1982

Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to
digitized documents strictly for personal use. Each copy of any part of this document must
contain these Terms of use.
Abstract: Generalizations of quasi-injective modules and p-injective modules over continuous rings, noted pQ(fQ)-injectivity, PQF-injectivity, MUP-injectivity, are introduced to study von Neumann regular rings. Sufficient conditions for von Neumann regularity are given. Left continuous regular, left Noetherian, quasi-Frobeniusian and semi-simple Artinian rings are characterized.

Key words: p-injective, f-injective, pQ-injective, fQ-injective, PQF-injective, MUP-injective, von Neumann regular, continuous regular, V-rings, CS-rings, Noetherian, quasi-Frobeniusian, semi-simple Artinian.

Classification: 16A12, 1A230, 16A32, 16A36, 16A40, 16A52.

Throughout, $A$ represents an associative ring with identity and $A$-modules are unitary. $Z$, $J$ will denote respectively the left singular ideal and the Jacobson radical of $A$. A von Neumann regular ring $A$ may be characterized by any one of the following conditions: (a) every left $A$-module is flat; (b) every left $A$-module is p-injective (f-injective). Note that if $I$ is a p-injective left ideal of $A$, then $A/I$ is a flat left $A$-module. As usual, an ideal of $A$ will always mean a two-sided ideal. $A$ is called fully idempotent (resp. fully left idempotent) if every ideal (resp. left ideal) of $A$ is idempotent. Following [3], $A$ is called a left V-ring if every simple left...
A-module is injective. It is now well-known that there is no inclusion between the classes of arbitrary von Neumann regular rings and V-rings [5]. However, they coincide in the commutative case (I. Kaplansky). Y. Utumi introduced left continuous rings as a generalization of left self-injective rings (cf. [9]). The notion of continuity was extended to modules and studied by various authors. Quasi-injective modules, intermediate between injective and continuous modules, have been extensively studied (cf. for example, the bibliography of [3], [4],[6]). Recall that (1) A left A-module M is quasi-injective iff any left A-homomorphism of every left submodule of M into M extends to an endomorphism of \( A^\cdot M \); (2) \( A^\cdot M \) is continuous iff (a) every complement left submodule of M is a direct summand of \( A^\cdot M \) and (b) every left submodule which is isomorphic to a direct summand of \( A^\cdot M \) is a direct summand of \( A^\cdot M \); (3) \( A^\cdot M \) is \( p \)-injective (\( f \)-injective) iff, for any principal (resp. finitely generated) left ideal I of A, any left A-homomorphism \( g: I \rightarrow M \), there exists \( y \in M \) such that \( g(b) = by \) for all \( b \in I \). It is easy to see that there is no inclusion between the classes of quasi-injective left modules and \( p \)-injective (\( f \)-injective) left modules.

We now introduce the following definitions.

**Definition 1.** A left A-module M is called MUP-injective if, for any complement left ideal C of A, a \( \in A \), any left A-homomorphism \( g: C a \rightarrow M \), there exists \( y \in M \) such that \( g(ca) = cay \) for all \( c \in C \).

**Definition 2.** A left A-module M is called pQ-injective (resp. (1) \( fQ \)-injective ; (2) \( pQF \)-injective) if, for any \( p \)-injective (resp. (1) \( f \)-injective; (2) projective) left submodule
N of M, any left A-homomorphism of N into M may be extended to an endomorphism of $A^M$.

MUP-injective left modules generalize injective left modules and $pQF$-injective modules over left continuous rings. $PQF$-injectivity, $pQ$-injectivity and $FQ$-injectivity generalize quasi-injectivity.

A is called a left MUP (resp. $pQ$, $fQ$, $PQF$)-injective ring if $A$ is MUP (resp. $pQ$, $fQ$, $PQF$)-injective.

Since a finitely generated $p$-injective left ideal of $A$ is a direct summand of $A^A$, then any left Noetherian ring is left $pQ$ and $fQ$-injective.

**Proposition 1.** If $A$ is semi-prime, then any simple left $A$-module is MUP-injective. Consequently, MUP-injectivity does not imply $p$-injectivity.

**Proof.** Let $M$ be a simple left $A$-module, $C$ a complement left ideal of $A$, $a \in A$, $g : Ca \rightarrow M$ a non-zero left $A$-monomorphism. Then $g$ is an isomorphism (since $A^M$ is simple) which implies that $Ca$ is a minimal left ideal of $A$. Since $A$ is semi-prime, $Ca = Ae$, where $e = e^2 \in A$. If $y = g(e) \in M$, then for any $a \in A$, $g(ae) = aeg(e) = ay$ which proves that $A^M$ is MUP-injective. Since simple modules over arbitrary semi-prime rings need not be $p$-injective (even in the commutative case), then the last assertion of Proposition 1 follows.

We see from Proposition 1 that MUP-injectivity does not imply $Tp$-injectivity considered in [17]. The converse is not true either (otherwise, by [17, Theorem 3] and Theorem 3 below, any simple regular ring would be left self-injective!). $MUP$-injective left modules need not be left continuous (cf. Theorems
3 and 10). Recall that a left $A$-module is non-singular iff its singular submodule is zero.

**Lemma 2.** Let $A$ be a left MUP-injective ring. Then

1. Any non-singular finitely generated left ideal of $A$ is a direct summand of $_A A$;
2. $A/Z$ is von Neumann regular and $Z = J$;
3. For any $a \in A$, $\ell(a) = 0$ iff $a$ is right invertible;
4. Every left or right $A$-module is divisible.

**Proof.** (1) Let $P = A b$, $a + b \in A$, be a non-singular principal left ideal. If $C$ is a non-zero complement left ideal such that $L = C \oplus \ell(b)$ is an essential left ideal, $g : C b \rightarrow A$ the left $A$-homomorphism defined by $g(c b) = c$ for all $c \in C$, then $g$ is a monomorphism, whence there exists $y \in A$ such that $c = g(c b) = cy$ for all $c \in C$. Then $C \subseteq \ell(b - byb)$ and therefore $L \subseteq \ell(b - byb)$ which yields $b = byb$ (since $A P$ is non-singular).

This proves that $A b$ is generated by an idempotent. Now if $F = A b + A a$ is non-singular, $b, a \in A$, then $A b = A e$, where $e = e^2 \in F$ and $F = A e + A a(1 - e) = A e + A w$, where $A a(1 - e) = A w$, $w = w^2 \in F$. If $v = (1 - e) w$, then $w v = w$, $v^2 = v$ and $A w = A v$ yielding $F = A e + A v = A(e + v)$ which is again generated by an idempotent. (1) then follows by induction on the number of generators.

(2) The proof of (1) shows that $A/Z$ is a von Neumann regular ring. Now if $z e Z$, $a e A$. $L = \ell(z a)$, then with $w = 1 - z a$, $L \cap \ell(w) = c$ implies $\ell(w) = 0$. If $g : A w \rightarrow A$ is the left $A$-homomorphism defined by $g(b w) = b$ for all $b \in A$, then there exists $v \in A$ such that $b = g(b w) = bwv$ for all $b \in A$. In particular, $l = w v = (1 - z a) v$ which implies that $z e J$. Then $J/Z$ is contained in the Jacobson radical of $A/Z$ which is zero, whence $Z = J$. - 430 -
(3) If $a \in A$ such that $ac = 1$ for some $c \in A$, then $L(a) = 0$. Conversely, let $L(a) = 0$. If $f : A \rightarrow A$ is the map $b \rightarrow b(a \in A)$, then there exists $d \in A$ such that $1 = f(a) = ad$.

(4) If $c$ is a non-zero-divisor of $A$, $cd = 1$ for some $d \in A$ by (3), and $c = dcd$ implies $dc = 1$. For any left $A$-module $M$, $M = cdM \subseteq cM \subseteq M$ which implies $M = cM$ and similarly, any right $A$-module is divisible.

Lemma 2 enables us to have a nice characterization of division rings. As usual, a left (right) ideal of $A$ is called reduced if it contains no non-zero nilpotent element.

**Corollary 2.1.** The following conditions are equivalent:

1. $A$ is a division ring;
2. $A$ is a prime left self-injective ring containing a non-zero reduced left ideal;
3. $A$ is a prime left MUP-injective ring containing a non-zero reduced left ideal.

**Proof.** Obviously, (1) implies (2) and (2) implies (3).

Assume (3). Let $I$ be a non-zero reduced left ideal, $e + b \in I$. Then $Aab$ is non-singular. By Lemma 2(1), $Ab$ is a direct summand of $A$ which implies that $A$ is an integral domain [13, Proposition 6]. Now by Lemma 2(3), any non-zero element of $A$ is right invertible which proves that (3) implies (1).

**Theorem 3.** The following conditions are equivalent:

1. $A$ is left continuous regular;
2. Every left $A$-module is MUP-injective;
3. Every essential left ideal of $A$ is MUP-injective;
4. Any ideal or complement left ideal of $A$ is a MUP-injective left $A$-module;
(5) A jg a left non-singular left MUP-injective ring whose complement left ideals are finitely generated.

Proof. Obviously, (1) implies (2) which, in turn, implies (3).

Assume (3). Let I be either a principal or a complement left ideal of A, K a complement left ideal such that \( L = I \oplus K \) is an essential left ideal. If \( f: I \rightarrow L \) is the natural injection, there exists \( u \in L \) such that \( f(b) = bu \) for all \( b \in I \). If \( u = c + k \), \( c \in I \), \( k \in K \), then \( b = b(c+k) \) implies \( b = bc \), whence \( c = c^2 \). Since \( I = Ac \), then \( I = Ac \) which proves \( A \) left continuous regular and (3) implies (4).

Assume (4). If \( z \in Z \), \( i: Az \rightarrow Z \) the canonical injection, then \( z = i(z) = zw \) for some \( w \in Z \). Now \( Az \cap \mathfrak{Z}(w) = 0 \) implies \( z = 0 \), whence \( Z = 0 \). By Lemma 2(2), \( A \) is von Neumann regular. If \( C \) is a complement left ideal, \( j:C \rightarrow C \) the identity map, there exists an idempotent \( e \in C \) such that \( C = Ae \). Thus (4) implies (5).

(5) implies (1) by Lemma 2(2).

The next result shows that arbitrary p-injective modules need not be pQ-injective.

Theorem 4. The following conditions are equivalent:

1. \( A \) is a left Noetherian ring whose p-injective left modules are injective;
2. Every p-injective left \( A \)-module is pQ-injective.

Proof. Obviously, (1) implies (2).

Assume (2). Let \( M \) be a p-injective left \( A \)-module, \( \mathfrak{H} M \) the injective hull of \( _A M \). Then \( B = _A M \otimes _A \mathfrak{H} \) is p-injective and therefore pQ-injective. If \( i:M \rightarrow B \) is the natural injection,
j:M→H and K:H B the inclusion maps, since _A_ is _PQ_-injective, there exists a left _A_-homomorphism _g:B→B_ such that _gkj = i_.

If _p:B→M_ is the natural projection, then _pgkj = p_1 = identity map on _M_. Thus _f = pgk_ is a left _A_-homomorphism of _H_ into _M_ such that _fj = identity map on _M_. This proves that _A^M_ is a direct summand of _A_H_, whence _M = H_ is injective. Since we know that any direct sum of injective left _A_-modules is _PQ_-injective and hence injective, which proves that _A_ is a left Noetherian [4, Theorem 20.1]. Thus (2) implies (1).

It is well-known that _A_ is left hereditary iff the sum of any two injective left _A_-modules is injective. Since a commutative ring _A_ is regular iff every simple _A_-module is _PQ_-injective, the next corollary then follows.

**Corollary 4.1.** Let _A_ be a commutative ring. Then

1. _A_ is Noetherian hereditary iff the sum of any two _PQ_-injective _A_-modules is _PQ_-injective and _PQ_-injective;
2. _A_ is semi-simple Artinian iff _PQ_-injective _A_-modules coincide with _PQ_-injective _A_-modules.

The next theorem may be similarly proved.

**Theorem 5.** The following conditions are equivalent:

1. _A_ is left Noetherian;
2. Every _PQ_-injective left _A_-module is injective;
3. Every _PQ_-injective left _A_-module is _PQ_-injective.

**Corollary 5.1.** If _A_ is a left _MUP_-injective ring such that every _PQ_-injective left _A_-module is _PQ_-injective, then _A_ is left Artinian.

**Proof.** By Lemma 2 (2) and Theorem 5, _A/J_ is semi-simple Artinian. Since _A_ is left Noetherian, then _J = Z_ is nilpotent.
which implies $A$ semi-primary. Therefore $A$ is left Artinian by [4, Proposition 9.12].

We are now in a position to give some new characteristic properties of quasi-Frobeniusean rings. Since it is well-known that $A$ is semi-simple Artinian iff every left $A$-module is quasi-injective, then the next result shows that PQF-injectivity effectively generalizes quasi-injectivity.

**Theorem 6.** The following conditions are equivalent:

1. $A$ is quasi-Frobeniusean;
2. Every left $A$-module is PQF-injective;
3. The direct sum of any projective left $A$-module and any injective left $A$-module is quasi-injective.
4. The direct sum of any projective left $A$-module and any injective left $A$-module is PQF-injective;
5. $A$ is a left $p$-injective ring such that every $p$-injective left $A$-module is quasi-injective.

**Proof.** Since $A$ is quasi-Frobeniusean iff every projective left $A$-module is injective [4, Theorem 24.20], then (1) implies (2) and (3).

Obviously, either (2) or (3) implies (4).

Assume (4). The proof of Theorem 4 shows that any projective left $A$-module is injective. Then (4) implies (5) by [4, Theorem 24.20].

(5) implies (1) by [4, Theorem 24.20] and Theorem 5.

Left WP-rings (weak $p$-injective) and left CPP-rings are considered in [14] and [12] respectively.

Applying [14, Lemma 1.1] and [12, Theorem 2], we get

**Proposition 7.** The following conditions are equivalent
for a left PQF-injective ring $A$:

1. $A$ is von Neumann regular;
2. $A$ is a left WP-ring;
3. $A$ is a left CPP-ring.

It may be noted that PQF-injectivity does not imply $p$-injectivity and the converse is not true either.

Lemma 2(2) implies the next result.

**Proposition 8.** (1) If $A$ is a left PQF-injective ring such that for any complement left ideal $C$ of $A$, $a \in A$, $A_a C$ is projective, then $A$ is a left MUP-injective regular ring.

(2) A primitive left MUP-injective ring is von Neumann regular.

(3) A MUP-injective left or right $V$-ring is von Neumann regular (cf. [5, Query (b)]).

(4) If $A$ is left TP-injective such that every complement
left ideal is an ideal, then $A/Z$ is von Neumann regular (this
extends [17, Theorem 9]).

At this point, we may note that there is no inclusion be­
tween the class of $p$-injective modules and any of the four
classes of modules which we have introduced at the beginning
of this note. We now consider a generalization of continuous
modules. Following [2], a left $A$-module $M$ is called CS if eve­
ry complement left submodule is a direct summand of $A^\vee$. Inde­
ed, CS-modules generalize even quasi-continuous modules effec­
tively (cf. [1]). If $A^\vee M$ is CS, then for complement left sub­
module $N$, $A^\vee N$ is CS [2, Proposition 2.2] and every left $A$-ho­
momorphism of $N$ into $M$ extends to an endomorphism of $A^\vee M$. But if

$A^\vee M$ is CS and $D$ a left submodule of $M$ which is CS, then $D$ is
not necessarily a complement submodule of $A^M$. This motivates the following class of rings: Write "A satisfies (§)" if, for any left $A$-module $M$, any CS left submodule $N$, every left $A$-homomorphism of $N$ into $M$ may be extended to an endomorphism of $A^M$.

Lemma 9. The following conditions are equivalent:

1. A is a left Noetherian left $V$-ring whose CS left modules are injective;
2. A satisfies (§).

Proof. (1) implies (2) evidently.

Assume (2). If $M$ is a CS left $A$-module, $A^H$ the injective hull of $A^M$, $B = A^M \oplus A^H$, then any left $A$-homomorphism of $M$ into $B$ extends to an endomorphism of $A^B$. The proof of Theorem 4 shows that $M = H$ is injective. Then A is a left Noetherian, left $V$-ring by [4, Corollary 20.3E].

ALD and left CM-rings are studied in [15] and [16]. Semi-prime rings whose left ideals are left annihilators must be semi-simple Artinian [10]. But semi-prime rings whose ideals are left annihilators are not necessarily Noetherian or regular. Left CM, left $V$-rings need not be regular (the Cozen's domains [4] are here relevant). Recall that $A$ is of bounded index iff the supremum of the indices of the nilpotent elements of $A$ is finite.

Theorem 10. The following conditions are equivalent:

1. A is semi-simple Artinian;
2. A is a left MUP-injective ring satisfying (§);
3. A is a right MUP-injective ring satisfying (§);
4. A is a left p-injective ring satisfying (§);
(5) A is a right p-injective ring satisfying (\(\star\));
(6) A is an ALP ring satisfying (\(\star\));
(7) A is a left CM, left PQF-injective ring such that for any ideal \(T\), any \(a \in A\), \(T_a\) is a projective left \(A\)-module;
(8) A is a semi-prime ring such that the direct sum of any non-zero projective left \(A\)-module and any non-zero injective left \(A\)-module is PQF-injective.
(9) The direct sum of a non-zero projective and a non-zero PQF-injective \(A\)-modules is quasi-injective;
(10) A is a semi-prime left CM, left MUP-injective ring whose ideals are left annihilators;
(11) A is a regular ring of bounded index such that every ideal is a left annihilator;
(12) A is a semi-prime ring whose ideals are left annihilators and whose simple factor rings are Artinian;
(13) A is a semi-prime left p-injective ring such that every p-injective left \(A\)-module is PQ-injective;
(14) A is a left PQF-injective ring whose cyclic left modules are either injective or projective.

Proof. Obviously, (1) implies (2) and (3).

Either (2) or (3) implies both (4) and (5) by Lemma 2(2) and Lemma 9.

Since semi-prime left p-injective rings with maximum condition on left annihilators are semi-simple Artinian, then either (4) or (5) implies (6) by Lemma 9.

(6) implies (7) by [15, Theorem 1.3].

Assume (7). Since every principal left ideal of \(A\) is projective, then \(Z = C\) and since \(A\) is left CM, then \(A\) is either reduced or semi-simple Artinian. Suppose that \(A\) is reduced.
Since \( A \) is a left PQF-injective ring, then \( A \) is left \( p \)-injective whose principal left ideals are projective which implies \( A \) regular and hence strongly regular. Therefore every left ideal of \( A \) is an ideal and is therefore a projective left \( A \)-module which implies \( A \) left hereditary, whence \( A \) is left self-injective. By a well-known result of B. Osofsky, \( A \) is semi-simple Artinian and (7) implies (8).

Since a semi-prime quasi-Frobeniusean ring is semi-simple Artinian, then (8) implies (9) by Theorem 6.

Since a simple left \( A \)-module is PQF-injective [4, Theorem 24.20] and the proof of Theorem 4 shows that (9) implies (10).

Assume (10). Since \( A \) is semi-prime whose ideals are left annihilators, then every ideal of \( A \) is generated by a central idempotent. This implies \( Z = 0 \) which yields \( A \) regular by Lemma 2(2). Since \( A \) is left CM, then \( A \) is either strongly regular or semi-simple Artinian. In either case, \( A \) has bounded index and (10) implies (11).

\( (11) \) implies (12) by [7, Corollary 7.10].

Assume (12). Then every ideal of \( A \) is generated by a central idempotent, whence \( A \) is a fully idempotent ring whose prime factor rings are simple. \( A \) is therefore regular by [7, Corollary 1.18]. Then any essential left ideal \( L \) of \( A \) contains an essential left ideal \( E \) which is an ideal of \( A \) [7, Lemma 6.20]. Now \( E \) is generated by a non-zero central idempotent which implies \( E = L = A \). This proves \( A \) semi-simple Artinian and therefore (12) implies (13).

(13) implies (14) by Theorem 4.

Finally, (14) implies (1) by [8, Corollary 10].

- 438 -
We now mention, without proof, an analogue for certain PQF-injective modules of a well-known theorem of C. Faith - Y. Utumi [6, Theorem 2.16] concerning quasi-injective modules.

**Theorem 11.** Let $M$ be a PQF-injective left $A$-module such that every complement left submodule is projective. If $E = \text{End}(A^M)$, then $E/J(E)$ is von Neumann regular, where $J(E) = \{f \in E/\ker f | f \text{ is essential in } A^M \}$ is the Jacobson radical of $E$.

Let us add a last result on MUP-injective rings.

**Proposition 12.** If $A$ is a prime left MUP-injective ring, then the centre of $A$ is a field.

**Proof.** Let $D$ denote the centre of $A$. If $d + yd \in D$, the proof of Lemma 2(1) shows that there exist an essential left ideal $L$ of $A$ and an element $y$ of $A$ such that $L \subseteq (d - yd)$. Since $d \in D$, $Ad(1 - yd) = L(d - yd) = 0$ implies $L \subseteq L(1 - yd)$, whence $1 - yd \in Z = J$ by Lemma 2(2). It follows that there exists $u \in A$ such that $uy = 1$. If $v = uy$, since $d \in D$, then $1 = vd = v^3d^3 = (d^2v^3)d$ and $vd^2 = d^2v = dvd = d$ which yields $d(d^2v^3)d = d$.

It is therefore sufficient to show that $d^2v^3 \in D$ and then $D$ will be a field. For any $a \in A$, $v(d^2a) = da = ad = a(d^2v) = (d^2a)v$ which yields $(d^2v^3)a = v^3(d^2a) = (d^2a)v^3 = a(d^2v^3)$. This proves that $d^2v^3 \in D$.

Simple rings, semi-simple Artinian rings and strongly regular rings are all biregular. It is known that (1) Von Neumann regular rings need not be biregular; (2) Biregular rings (even if reduced) need not be regular; (3) Simple Noetherian rings need not be Artinian. We know that biregular rings are fully left and right idempotent. We conclude with a few remarks, the first two being a sequel to [17].

- 439 -
Remark 1. If $A$ is a biregular ring which is of left finite Goldie dimension, then (1) Every ideal of $A$ is generated by a central idempotent; (2) $A$ is a finite direct sum of simple rings; (3) $A$ is the only essential left (or right) ideal which is an ideal of $A$.

Remark 2. If $A$ is left non-singular left CM-ring whose ideals are left annihilators, then $A$ is a Baer, Dedekind finite left and right CS, biregular ring.

Remark 3. If $A$ is a left self-injective ring whose essential left ideals are left annihilators, then every left ideal of $A$ is a left annihilator. In that case, each semi-prime factor ring of $A$ is semi-simple Artinian.

Remark 4. (1) If $A$ is a right MUP-injective ring, then (a) Any reduced finitely generated right ideal is generated by an idempotent. Consequently, [1, Theorem 12] holds for right MUP-injective rings whose complement right ideals are finitely generated; (b) If every maximal essential right ideal of $A$ is an ideal, then $A$ is von Neumann regular iff every simple left $A$-module is flat (this extends [11, Proposition 2.2(IV)]); (2) $A$ is strongly regular iff $A$ is a semi-prime left duo ring whose MUP-injective left modules are p-injective (apply [15, Theorem 1.3] and Proposition 1).

Remark 5. Let $A$ be a left PQF-injective ring. Then (1) Every left and right $A$-module is divisible; (2) If $A$ is left uniform, then $A/Z$ is von Neumann regular and $Z = J$; (3) $A$ is left self-injective regular iff $A$ is left non-singular such that every finitely generated non-singular left $A$-module is
projective (apply [6, Theorem 3.12]); (4) A is simple Artini-an iff A is prime left non-singular left CM.

**Remark 6.** A is a left principal ideal ring iff every finitely generated left ideal of A is principal and every p-injective left A-module is pQ-injective (cf. Theorem 4).

In view of [15, Theorem 1.7] and [18, Theorem 2.21], we raise the following question: Is A strongly regular if every maximal left ideal of A is an ideal and every simple left A-module is flat?

**References**


Université Paris VII, U.E.R. de Mathématiques, 2, Place Jussieu, 75251 Paris Cedex 05, France

(Oblatům 4.2. 1982)