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FIXED POINTS FOR GENERALIZED NONEXPANSIVE MAPPINGS

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Abstract: The aim of the present paper is to prove the existence of fixed points for generalized nonexpansive mappings in convex metric spaces. Such spaces, introduced by Takahashi, included Banach spaces and results of this paper generalize those of Dotson, Rhoades and others.

Key words and phrases: Convex metric spaces, star-shaped metric spaces, normal structure, nonexpansive, generalized nonexpansive mappings, quasi-contraction, locally contractive, uniformly locally contractive mappings, fixed points.

Classification: 47H10, 52H25

Introduction. In 1970 Takahashi [13] introduced a notion of convexity in metric spaces (see Definition 1.1) and extended some fixed point theorems to convex metric spaces. Subsequently Itoh [5], Machado [7], Tallman [12], Naimpally and Singh [8], Naimpally, Singh and Whitfield [9] studied convex metric spaces and fixed point theorems. This paper is a continuation of these investigations.

In section 1 a fixed point theorem is proved for generalized nonexpansive mappings. Section 2 deals with the existence of fixed points in nonconvex domains.

Throughout this paper X and I denote, respectively, a metric space X, with a metric d, and the unit interval [0,1].

The following definition was introduced by Takahashi [13].

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Definition 1.1. Let X be a metric space. A mapping $W: X \times X \times I \longrightarrow X$ is said to be a convex structure if, for each $x, y \in X$ and $\lambda \in I$, the following condition is satisfied:

$$d(u, W(x,y,\lambda)) \leq \lambda d(u,x) + (1-\lambda)d(u,y)$$

for all $u \in X$. A metric space with a convex structure will be called a convex metric space.

If X is a Banach space, then, as a metric space with $d(x,y) = \|x-y\|$, the mapping $W: X \times X \times I \longrightarrow X$ defined by $W(x,y,\lambda) = \lambda x + (1-\lambda)y$ is a convex structure. Thus a Banach space, or any convex subset of a Banach space, is a convex metric space. There are many convex metric spaces which cannot be imbedded in any Banach space [13, Examples 1.1 and 1.2].

Definition 1.2. Let X be a convex metric space. Let K be a nonempty subset of X. K is called convex if $W(x,y,\lambda)$ belongs to K for all $x, y \in K$ and $\lambda \in I$.

Definition 1.3. A metric space X is said to be star-shaped if there exists an $x_0 \in X$ and a mapping $W: X \times \{x_0\} \times I \longrightarrow X$ such that, for each $x, y \in X$ and $\lambda \in I$,

$$d(x, W(y,x_0,\lambda)) \leq \lambda d(x,y) + (1-\lambda)d(x,x_0).$$

Star-shaped metric spaces are generalizations of star-shaped subsets of Banach spaces, where a subset K of a Banach space is star-shaped if there exists an $x_0 \in K$ such that for each $x \in K$, $\lambda \in I$, $\lambda x + (1-\lambda)x_0 \in K$. It is obvious that convex metric spaces are star-shaped metric spaces, but not conversely.

Definition 1.4. A star-shaped metric space X is said to satisfy condition (I) if for any $x, y \in X$, $\lambda \in I$,

$$d(W(x,x_0,\lambda), W(y,x_0,\lambda)) \leq \lambda d(x,y).$$

The condition (I) is always satisfied in any normed linear space. Indeed, let $W(x,x_0,\lambda) = \lambda x + (1-\lambda)x_0$ and $W(y,x_0,\lambda) = \lambda y + (1-\lambda)x_0$. Then $d(W(x,x_0,\lambda), W(y,x_0,\lambda)) = \|W(x,x_0,\lambda) - W(y,x_0,\lambda)\| = \|\lambda x - \lambda y\| = \lambda\|x - y\| = \lambda d(x,y)$.

Definition 1.5. A mapping $T:X \longrightarrow X$ is said to be nonexpansive if $d(Tx,Ty) \leq d(x,y)$ for all $x,y \in X$. T is called generalized nonexpansive if for all $x,y \in X$, $d(Tx,Ty) \leq \max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}$. T is called a quasi-contraction if there exists a constant k, $0 \leq k < 1$, such that for all $x,y \in X$, $d(Tx,Ty) \leq k \max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}$.

A generalized nonexpansive mapping may or may not be continuous. Indeed, let $X = [0,5]$ with the usual metric. Define $T:X \longrightarrow X$ by $T(x) = x/2$, $0 \leq x < 4$ and $T(x) = -2x + 10$, $4 \leq x \leq 5$. Then T is a continuous generalized nonexpansive mapping, which is not nonexpansive (take $x = 4$, $y = 5$). On the other hand, if $X = [0,1]$, and $T(x) = 0$, $0 \leq x \leq 1/2$, $T(x) = 1/2$, $1/2 < x \leq 1$, then T is a discontinuous generalized nonexpansive mapping.

The following is proved in [10, Theorem 2].

Theorem R. Let X be a complete reflexive Banach space, K a closed, bounded, convex subset of C, T a selfmap of K satisfying

$$\|Tx-Ty\| \leq a(x,y)\,\|x-y\| + b(x,y)\,\|x-Tx\| + b(y,x)\,\|y-Ty\| + c(x,y)\,\|x-Ty\| + c(y,x)\,\|y-Tx\|,$$

where a, b, c are nonnegative real valued mappings from $K \times K$ satisfying $(a+b+c)(x,y) + (b+c)(y,x) \leq 1$ for all $x,y \in K$. If in addition, $\sup_{x,y \in K}(b(x,y) + c(y,x)) < 1$.

Then T has a fixed point in K.

In Theorem R, the mapping T is not necessarily continuous.

A theorem similar to Theorem R, for discontinuous generalized nonexpansive mappings is not possible even in uniformly convex Banach spaces, as can be seen from the following example.

Example 1.1. Let $X = R$ and $K = [0,1] \subset R$. Define $T:K \longrightarrow K$ by $T(x) = 3/4$, $0 \leq x \leq 1/2$, $T(x) = 1/2$, $1/2 < x \leq 1$. Clearly T is a generalized nonexpansive mapping without a fixed point.

However, for continuous generalized nonexpansive mappings the following can be proved.

Theorem 1.1. Let X be a compact star-shaped metric space satisfying condition (I). Let $T:X \longrightarrow X$ be a continuous generalized nonexpansive mapping. Then T has a fixed point.

Proof. Let $x_0 \in X$. For $0 < k < 1$, define the mapping T_k as follows: $T_k(x) = W(Tx, x_0, k)$. Since X is star-shaped, it follows that T_k maps X into itself. Using condition (I), T_k is a quasi-contraction. Indeed, $d(T_k(x), T_k(y)) = d(W(Tx,x_0,k), W(Ty,x_0,k)) \leq kd(Tx,Ty) \leq k \max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}$ for all $x,y \in X$. It follows from [2, Theorem 1] that there exists an $x_k \in X$ such that $T_k(x_k) = x_k$. Moreover,

$$d(T(x_k),x_k) = d(Tx_k), T_k(x_k)) = d(T(x_k), W(Tx_k,x_0,k))$$

$$\leq kd(T(x_k), T(x_k)) + (1-k)d(T(x_k),x_0),$$

which approaches zero as $k \longrightarrow 1$, since X is bounded. Hence $\lim_k d(T(x_k),x_k) = 0$. Since X is compact, $\{x_k\}$ has a convergent subsequence $\{x_{k_i}\}$ converging to some point $x \in X$. Thus $\lim_i d(x_{k_i}, T(x_{k_i})) = 0$. By the continuity of T, it follows that $T(x_{k_i}) \longrightarrow T(x)$. From the triangular inequality, $d(x,Tx) \leq$ $\leq d(x,x_{k_i}) + d(x_{k_i},T(x_{k_i})) + d(T(x_{k_i}),T(x))$. Taking the limit as $i \longrightarrow \infty$ yields $x = Tx$.

An immediate consequence of Theorem 1.1 is the following.

Corollary 1.1. [3, Theorem 1.] Let X be a Banach space and K be a nonempty compact star-shaped subset of X. Let $T:K \longrightarrow K$ be a nonexpansive mapping. Then T has a fixed point in K.

Definition 2.1. A mapping T of X into itself is said to be locally contractive if for each $x \in X$ there exist ε and λ ($\varepsilon >$ > 0, $0 \leq \lambda < 1$), which may depend on x, such that $p,q \in S(x,\varepsilon) =$ $= \{y:d(x,y) < \varepsilon\}$ implies $d(T(p),T(q)) \leq \lambda d(p,q)$. T is called $(\varepsilon-\lambda)$-uniformly locally contractive if it is locally contractive and both ε and λ are independent of x.

Definition 2.2. A mapping T of X into itself is called a Banach operator if there exists a constant k, $0 \leq k < 1$, such that for all $x \in X$, $d(T^2(x),T(x)) \leq kd(Tx,x)$.

A Banach operator may not be continuous, and may have more than one fixed point. The following lemma, whose easy proof is omitted, will be needed.

Lemma 2.1. Let X be a complete metric space, $T:X \longrightarrow X$ a continuous Banach operator. Then T has a fixed point.

Definition 2.3. Let X be a convex metric space. X satisfies condition (II) if for all $x,y,z \in X$ and $\lambda \in I$,

$$d(W(x,z,\lambda), W(y,z,\lambda)) \leq \lambda d(x,y).$$

Condition (II) is always satisfied in any normed linear space.

<u>Theorem 2.1</u>. Let X be a convex metric space satisfying condition (II). Let K be a compact subset of X. Let $T:K \rightarrow K$ be a continuous mapping. Suppose,

(i) there exists $q \in K$ and a fixed sequence of positive reals $k_n (k_n < 1)$ converging to 1, such that $W(Tx,q,k_n) \in K$ for all $x \in K$; further for each $x \in K$ and k_n, $d(T(W(Tx,q,k_n)),T(x)) \leq$ $\leq d(W(Tx,q,k_n,x)$,

or

(ii) K is star-shaped and $d(Tx,Ty) \leq d(x,y)$ whenever $d(x,y) < \varepsilon$ $(x,y \in K)$ for $\varepsilon > 0$. Then T has a fixed point.

<u>Proof</u>. Define the map T_n by

$$T_n(x) = W(Tx,q,k_n).$$

Then each T_n is a continuous Banach operator. Indeed, using condition (i) and (II),

$$\begin{aligned} d(T_n^2(x),\ T_n(x)) &= d(T_n(W(Tx,q,k_n)),\ W(Tx,q,k_n)) \\ &= d(W(T(W(Tx,q,k_n)),q,k_n)),\ W(Tx,q,k_n)) \\ &\leq k_n d(T(W(Tx,q,k_n),Tx) \\ &\leq k_n d(W(Tx,q,k_n),x) \\ &= k_n d(T_n(x),x). \end{aligned}$$

By hypothesis it follows that each T_n maps K into itself. It follows from Lemma 2.1 that there exists a $y_n \in K$ such that $T_n(y_n) = y_n$.

$$d(T(y_n),y_n) = d(T(y_n),\ W(Ty_n,w,k_n)) \leq$$

$$\leq k_n d(T(y_n), T(y_n)) + (1-k_n)d(T(y_n),q)$$

which approaches zero as $k_n \longrightarrow 1$. Hence $\lim_n d(y_n,T(y_n)) = 0$. Since K is compact, $\{y_n\}$ has a convergent subsequence $\{y_{n_i}\}$ converging to some point $y \in X$ and, from the above, $d(y_{n_i},T(y_{n_i})) \longrightarrow 0$. By the continuity of T it follows that $T(y_{n_i}) \longrightarrow T(y)$. Finally,

$$d(y,Ty) \leq d(y,y_{n_i}) + d(y_{n_i},T(y_{n_i})) + d(T(y_{n_i}),T(y)).$$

Taking the limit as $i \longrightarrow \infty$ yields $y = T(y)$.

If (ii) holds, then each map T_n defined by $T_n(x) = W(Tx,q,k_n)$ is (ε,k_n)-uniformly locally contractive. Moreover, K being compact and star-shaped is complete and ε-chainable. It follows from [4, Theorem 5.2] that each T_n has a fixed point y_n. The rest of the proof is the same as in (i).

Remark 2.1. If X is a Banach space and K is a star-shaped subset of X (in particular if K is convex) then, for a nonexpansive self mapping T of K, $(1-k_n)q + k_nT(x) \in K$, for each $x \in K$ and any sequence $\{k_n\}$ converging to $1 (k_n < 1)$, where q is the star center of K. Moreover, from the nonexpansiveness of T it follows that $\|T((1-k_n)q + k_nT(x)) - T(x)\| \leq \|(1-k_n)q + k_nTx - x\|$.

Remark 2.2. If K is star-shaped, then Theorem 1 of [3] follows from Theorem 2.1. Hypothesis (i) above weakens the star-shaped assumption, as can be seen from the following example.

Example 2.1. Let K be the set $\{(0,y): y \in [-1,1]\} \cup \{(1-\frac{1}{n},0): n \in N\} \cup \{(1,0)\}$ with the metric induced by the norm $\|(x,y)\| = |x| + |y|$. Define the map $T:K \longrightarrow K$ as follows: $T(0,y) = (0,-y)$, $T(1-\frac{1}{n},0) = (0,1-\frac{1}{n})$, $T(1,0) = (0,1)$. Then T satisfies condition (i) of Theorem 2.1 with the choice $q = (0,0)$, $k_n = 1 - \frac{1}{n}$, $n = 1,2,\ldots$, although K is not star-shaped.

Remark 2.3. Let X be a reflexive Banach space and K be a nonempty weakly compact convex subset of X having normal structure. It is clear from Example 1.1 that a discontinuous generalized nonexpansive selfmap of K may not have a fixed point. Also, if X is a Banach space and K is a nonempty weakly compact convex subset of X, then a continuous generalized nonexpansive map need not have a fixed point (see, e.g., [1] and [11]). It is known (6, Exercises 1.12, 1.13, 1.14) that a continuous generalized nonexpansive selfmap of a closed, bounded, convex subset of certain Banach spaces is fixed point free.

Problem. Must a continuous generalized nonexpansive mapping of a weakly compact convex subset of a reflexive Banach space with normal structure have a fixed point?

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