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MODEL-THEORETIC PROPERTIES OF CAUSE-AND-EFFECT STRUCTURES

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Abstract: Some questions of axiomatizability and decidability connected with the study of so-called cause-and-effect structures (as introduced by me under the influence of von Wright) are treated.

Key words: Causality relation, axiomatizability, decidability.

Classification: 03A05, 03B25, 03C20

Let a cause-and-effect structure be defined as follows. The domain consists of a set T of moments and a set S of states; the elements of $T\times S$ are called events. As relations and functions we have a chronological order $<\subseteq T\times T$, a time addition $+:T\times T\longrightarrow T$, a possibility of events $\Diamond\subseteq T\times S$, an actuality of events $\Box\subseteq T\times S$, and a cause-and-effect relation $\longmapsto\subseteq T\times S\dot{\times} T\times S$ (we write $t,s\longmapsto t',s'$ instead of $\longmapsto(t,s,t',s')$). The axioms we assume to be fulfilled by cause-and-effect structures are

(1) $\langle T,<,+\rangle$ is an ordered abelian group

(2) $\forall t\exists s\ \Diamond(t,s)$

(3) $\forall t\exists !!\ \Box(t,s)$

(4) $\forall t,s(\Box(t,s)\longrightarrow\Diamond(t,s))$

(5) $\forall t_1,s_1,t_2,s_2((t_1,s_1\longmapsto t_2,s_2)\wedge\Box(t_1,s_1)\longrightarrow\Box(t_2,s_2))$

(6) $\forall t_1, s_1, t_2, s_2((t_1, s_1 \mapsto t_2, s_2) \longrightarrow t_1 < t_2)$

(7) $\forall t_1, s_1, t_2, s_2, t(((t_1, s_1 \mapsto t_2, s_2) \longleftrightarrow$

$\longleftrightarrow (t_1 + t, s_1 \mapsto t_2 + t, s))$

(8) $\forall t, s(\Diamond (t, s) \longrightarrow \exists t', s'(\Diamond (t', s') \wedge (t', s' \mapsto t, s)))$.

Let CES denote the class of all cause-and-effect structures.

Cause-and-effect structures differ from causality structures as introduced (under inspiration of [1]) in [2] in that the axiom (5) of [2] constating, intuitively spoken, that "the behaviour of the system in the past is uniquely determined" is missing.

Given $\mathfrak{A} = \langle T \cup S, <, +, \Diamond, \mapsto, \Box \rangle \in$ CES, there is a natural way of embedding $\mathfrak{A}$ into a causality structure $\mathfrak{A}'$ by proceeding as follows. Let $\mathfrak{A}' = \langle (T \times T) \cup (S \cup \{s_0\}), <', +', \Diamond',$ $\mapsto', \Box' \rangle$, where

$<' = \{\langle\langle t_1, t_2\rangle, \langle t_3, t_4\rangle\rangle : t_1 < t_3 \vee (t_1 = t_3 \wedge t_2 < t_4)\}$

$+' = \{\langle\langle t_1, t_2\rangle, \langle t_3, t_4\rangle, \langle t_5, t_6\rangle\rangle : t_1 + t_3 = t_5 \wedge t_2 + t_4 = t_6\}$

$\Diamond' = \{\langle\langle t_1, t_2\rangle, s\rangle : (t_1 = 0 \wedge \Diamond(t_2, s)) \vee (t_1 \neq 0 \wedge s = s_0)\}$

$\mapsto' = \{\langle\langle t_1, t_1'\rangle, s_1, \langle t_2, t_2'\rangle, s_2\rangle : (t_1 = t_2 = 0 \wedge t', s \mapsto t', s) \vee (t_1 \neq 0 \wedge t_2 \neq 0 \wedge t_1 < t_2)\}$

$\Box' = \{\langle\langle t_1, t_2\rangle, s\rangle : (t_1 = 0 \wedge \Box(t_2, s)) \vee (t_1 \neq 0 \wedge s = s_0)\}$.

$\Box'$ is obtained from $\mathfrak{A}$ by adding a one-state (and, hence, uniquely determined) "past" which precedes the whole "world" $\mathfrak{A}$ and (in order to secure (1)) a one-state "future" (the same state as in the past) which follows the whole "world" $\mathfrak{A}$. Of course, the "metatheoretical complicatedness" of $\mathfrak{A}'$ is not exceeding that of $\mathfrak{A}$ although the technical treatment of $\mathfrak{A}'$

may be more labourious than that of $\mathfrak{A}$. This gives motivation to investigate the model-theoretic properties of causality structures by investigating the model-theoretic properties of cause-and-effect structures.

Let $\mathfrak{A} = \langle T \cup S, <, +, \Diamond, \mapsto, \Box \rangle \in CES$ be called <u>proper</u>, if, for every $\langle t,s \rangle \in \Diamond$, there is $\Box' \subseteq T \times S \times T \times S$ such that $\langle t,s \rangle \in \Box'$ and, likewise, $\mathfrak{A}' = \langle T \cup S, <, +, \Diamond, \mapsto, \Box' \rangle \in CES$. The class of proper cause-and-effect structures will be denoted by PCES.

<u>Theorem 1</u>: With respect to the signature $< , +, \Diamond, \mapsto, \Box >$, PCES is not EC.

Proof. We demonstrate CES$\setminus$PCES not to be closed under the operation of taking ultraproducts.

Let $\mathfrak{A}_n = \langle (\omega^* + \omega) \cup S_n, <, +, \Diamond_n, \mapsto_n, \Box_n \rangle$ $(n \in \omega)$ be defined as follows:

1. $\langle \omega^* + \omega, <, + \rangle$ is isomorphic to the additive group of integers
2. $S_n = \{0,1,2,3\} \times (\omega^* + \omega)$
3. $\Diamond_n = \{\langle x, \langle 0,x \rangle\rangle : x \equiv 0(2) \wedge x \leq 2n\}$
   $\cup \{\langle x, \langle 1,x \rangle\rangle : x \equiv 1(2) \wedge x \leq 2n + 1\}$
   $\cup \{\langle x, \langle 2,x \rangle\rangle : x \leq 0 \vee (x \equiv 0(2) \wedge x \leq 2n + 2)\}$
   $\cup \{\langle x, \langle 3,x \rangle\rangle : x > 1 \wedge (x \equiv 1(2) \vee x \geq 2n + 1)\}$
4. $\mapsto_n = \Diamond_n^2 \cap \{\langle x, \langle y,x \rangle, x', \langle y',x' \rangle\rangle :$
   $: (y=y'=3 \wedge x' = x+2 \wedge 0 < x \leq 2n+1)$
   $\vee (y=y'=3 \wedge x'=x+1 \wedge x \geq 2n+1)$
   $\vee (y=y'=2 \wedge x'=x+1 \wedge x < 0)$
   $\vee (y=y'=2 \wedge x'=x+2 \wedge 0 \leq x)$

$$\vee(y=y'=1 \wedge x'=x+2)$$
$$\vee(y=y'=0 \wedge x'=x+2$$
$$\vee(\langle x,y\rangle=\langle 0,2\rangle \wedge \langle x',y'\rangle = \langle 1,3\rangle)$$
$$\vee(\langle x,y\rangle=\langle 2n+1,1\rangle \wedge \langle x',y'\rangle = \langle 2n+2,2\rangle)$$
$$\vee(\langle x,y\rangle=\langle 2n+2,2\rangle \wedge \langle x',y'\rangle = \langle 2n+3,3\rangle)$$
$$\vee(\langle x,y\rangle=\langle 2n,0\rangle \wedge \langle x',y'\rangle = \langle 2n+3,3\rangle)\}$$

5. $$\Box_n = \Diamond_n \cap (\{\langle x,\langle 0,x\rangle : x < 2n\} \cup$$
$$\{\langle\langle x,\langle 1,x\rangle\rangle : x < 2n+1\} \cup$$
$$\{\langle x,\langle 2,x\rangle\rangle : x = 2n+2\} \cup$$
$$\{\langle x,\langle 3,x\rangle\rangle : x > 2n+2\}).$$

$\mathfrak{A}_3$ is illustrated by fig. 1 ($\Box_3$ cannot be taken from the figure itself, but this does not matter).

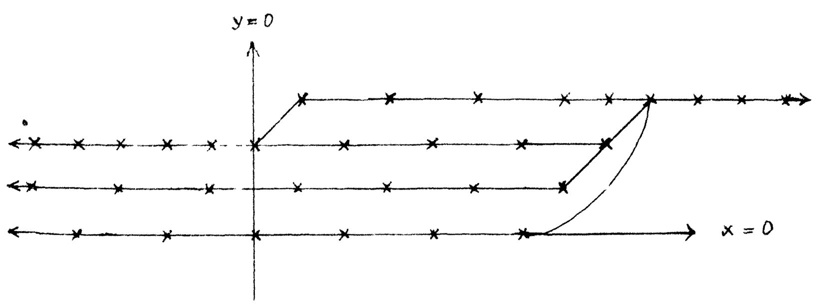

Fig. 1

In this figure, the event $\langle x,\langle y,x\rangle\rangle$ is marked by a cross at the point $\langle x,y\rangle$, and two crosses are connected by an arc if and only if the corresponding events are in cause-and-effect relation (Note that $\mapsto$ is not transitive!). It is easy to check that each $\mathfrak{A}_n$ is CES (the validity of (5) is based on the fact that states belonging to differenct events are different - a fact which cannot be taken from fig. 1 because $\langle x,\langle y,x\rangle\rangle$ is simply coded by $\langle x,y\rangle$; the validity of the remaining axioms can immediately be seen). On the other hand, no $\mathfrak{A}_n$ is PCES. For instance, there is no $\mathfrak{A}_n' = \langle\{\omega^* + \omega\} \cup S, <, +, \Diamond_n, \mapsto_n, \Box_n'\rangle \in$ CES such that $\langle 2,\langle 0,2\rangle\rangle \in \Box_n'$. For, assuming $\langle 0,\langle 2,0\rangle\rangle \in \Box_n'$, we have, by (5): $\langle 1,\langle 3,1\rangle\rangle, \langle 3,\langle 3,3\rangle\rangle, \ldots$ $\ldots, \langle 2n+1,\langle 3,2n+1\rangle\rangle, \langle 2n+2,\langle 3,2n+2\rangle\rangle \in \Box_n'$, and, again by (5), $\langle 2,\langle 2,2\rangle\rangle, \langle 4,\langle 2,4\rangle\rangle, \ldots, \langle 2n+2,\langle 2,2n+2\rangle\rangle \in \Box_n'$, but $\langle 2n+2,\langle 3,2n+1\rangle\rangle \in \Box_n'$, $\langle 2n+2,\langle 2,2n+2\rangle\rangle \in \Box_n'$ is in contradiction with (3).

Next we show that $\prod_{n \in \omega} \mathfrak{A}_n / \mathfrak{U} \in$ PCES, where $\mathfrak{U}$ is a non-principal ultrafilter over ω.

Let us investigate the structure $\prod_{n \in \omega} \mathfrak{A}_n / \mathfrak{U}$. The order is of type $(\omega^* + \omega) \cdot (\tau^* + \tau)$, so that the moments can be coded by couples $\langle \pm \alpha, n\rangle$, where $\alpha \in \tau$, $n \in \omega^* + \omega$. The substructure induced by all events possible in moments of type $\langle 0,n\rangle$ is illustrated by fig. 2:

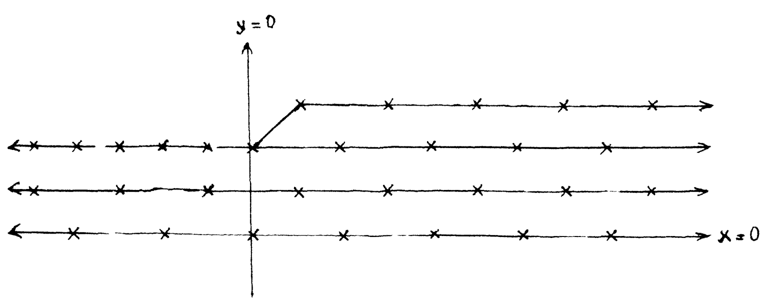

Fig. 2

The substructure induced by all events possible in moments $\langle \alpha_0, n \rangle$ where $\langle \alpha_0, n_0 \rangle$ is the moment attached to the event p of $\prod_{n \in \omega} \mathfrak{A}_n / \mathfrak{U}$ represented by the sequence $\{\langle 2n, \langle 2, 2n \rangle \rangle\}_{n \in \omega}$ is illustrated by fig. 3:

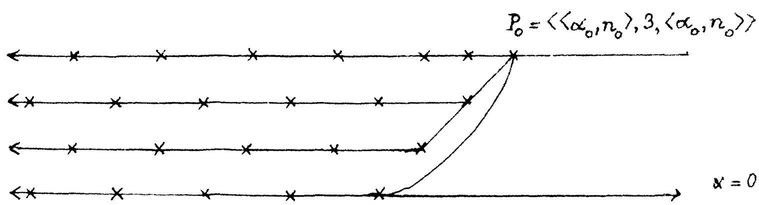

Fig. 3

For moments resting, the corresponding substructures are illustrated by fig. 4 ($\langle \alpha, 0 \rangle < 0$), fig. 5 ($0 < \langle \alpha, 0 \rangle < \langle \alpha_0, 0 \rangle$) and fig. 6 ($\langle \alpha, 0 \rangle > \langle \alpha_0, 0 \rangle$):

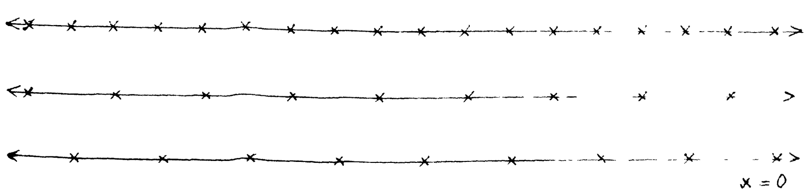

Fig. 4

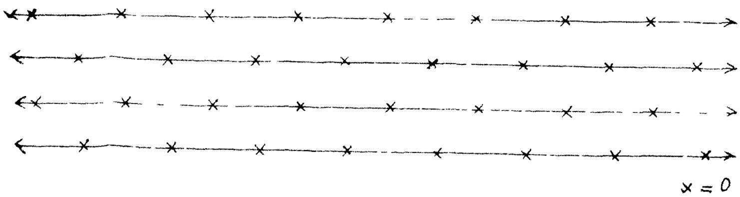

Fig. 5

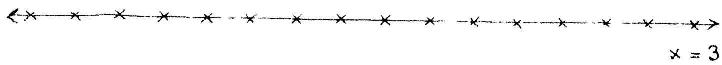

Fig. 6

For perspicuity: the difference between the $\mathfrak{A}_n$'s and $\prod_{n \in \omega} \mathfrak{A}_n / \mathfrak{U}$ consists in that the "distance" of the "critical points" in $\mathfrak{A}_n$ - $\langle 0, \langle 2,0 \rangle \rangle$ on one side and $\langle 2n+1, \langle 3, 2n+1 \rangle \rangle$, $\langle 2n+2, \langle 3, 2n+2 \rangle \rangle$, $\langle 2n+3, \langle 3, 2n+3 \rangle \rangle$, $\langle 2n+2, \langle 2, 2n+2 \rangle \rangle$ on the other side - has become infinite in $\prod_{n \in \omega} \mathfrak{A}_n / \mathfrak{U}$.

If $\alpha_1 \neq \alpha_2$, then the substructures induced by the moments of type $\langle \alpha_1, n \rangle$ and $\langle \alpha_2, n \rangle$, respectively, are, with respect to $\mapsto$, "not in contact" with one another. This enables us to solve the problem of finding an alternative actuality relation for an arbitrarily given event $\langle t,s \rangle$ by restricting ourselves to the substructure to which $\langle t,s \rangle$ belongs.

In case $\langle t,s \rangle$ belongs to a substructure as described in fig. 2, fig. 4, fig. 5, fig. 6, the problem is trivial. For the substructure described in fig. 3, we find the following possible alternative actuality relations ($\Box_1'$ coincides with that of the original actuality relation in $\prod_{n \in \omega} \mathfrak{A}_n / \mathfrak{U}$; $\Diamond$ is the possibility of events in the very same $\prod_{n \in \omega} \mathfrak{A}_n / \mathfrak{U}$):

$$\Box_1' = \Diamond \cap (\{\langle\langle \alpha_0, n\rangle, \langle 0, \langle \alpha_0, n\rangle\rangle\rangle : n < n_0 - 1\} \cup \{\langle\langle \alpha_0, n\rangle, \langle 1, \langle \alpha_0, n\rangle\rangle\rangle : n < n_0 - 1\} \cup \{\langle\langle \alpha_0, n_0 - 1\rangle, \langle 2, \langle \alpha_0, n_0 - 1\rangle\rangle\rangle\} \cup \{\langle\langle \alpha_0, n\rangle, \langle 3, \langle \alpha_0, n\rangle\rangle\rangle : n \geq n_0 \})$$

$$\Box_2' = \Diamond \cap (\{\langle\langle \alpha_0, n\rangle, \langle 0, \langle \alpha_0, n\rangle\rangle\rangle : n < n_0 - 2\} \cup \{\langle\langle \alpha_0, n\rangle, \langle 3, \langle \alpha_0, n\rangle\rangle\rangle : n \geq n_0 - 2\})$$

$$\Box_3' = \Diamond \cap (\{\langle\langle \alpha_0, n\rangle, \langle 1, \langle \alpha_0, n\rangle\rangle\rangle : n < n_0\} \cup \{\langle\langle \alpha_0, n\rangle, \langle 2, \langle \alpha_0, n\rangle\rangle\rangle : n < n_0\} \cup \{\langle\langle \alpha_0, n\rangle, \langle 3, \langle \alpha_0, n\rangle\rangle\rangle : n \geq n_0\}).$$

Each event in fig. 3 belongs to either $\Box_1'$, $\Box_2'$ or $\Box_3'$; thus, $\prod_{n \in \omega} \mathfrak{A}_n / \mathfrak{U}$ is proved to belong to PCES. ■

Let $\mathfrak{A}_n'$ be the causality structure obtained from $\mathfrak{A}_n$ by the method described above. Clearly, the $\mathfrak{A}_n'$ are not PCS in the sense of [2]. By using the same arguments as in the proof of Theorem 1 it can be shown that $\prod_{n \in \omega} \mathfrak{A}_n / \mathfrak{U}$ is PCS. Thus, we have: PCS is not EC, as already announced in [2].

A structure $\langle T \cup S, <, +, \Diamond, \mapsto \rangle$ may be called <u>actualizable law structure</u> whenever there is $\Box \subseteq \Diamond$ such that $\langle T \cup S, <, +, \Diamond, \mapsto, \Box \rangle$ is CES. The class of actualizable cause-and-effect structures is denoted by ALS. $\langle T \cup S, <, +, \Diamond, \mapsto \rangle$ may be called <u>universally actualizable law structure</u> whenever, for every $\langle t,s\rangle \in T\times S$, there is a $\Box \subseteq \Diamond$ containing $\langle t,s\rangle$ such that $\langle T\cup S, <, +, \Diamond, \mapsto, \Box\rangle$ is CES (the latter-mentioned structure is, then, automatically PCES). The class of all universally actualizable law structures may be denoted by UALS. It may be remarked here that the terminus "law structure" is motivated by the imagination that $\langle T\cup S, <, +, \Diamond, \mapsto\rangle$ describes the "physical laws" of our "world" $\langle T\cup S, <, +, \Diamond, \mapsto, \Box\rangle$, compare [2].

By "forgetting" the $\Box_n$ in the proof of Theorem 1, we get at once

<u>Theorem 2</u>: The class UALS is, with respect to the signature $\langle <, +, \Diamond, \mapsto\rangle$, not relatively finitizable to ALS (i.e. there is no φ such that UALS = ALS $\cap$ Mod($\{\varphi\}$)).

<u>Theorem 3</u>: With respect to the signature $\langle <, +, \Diamond, \mapsto\rangle$, ALS is not EC.

Proof. We demonstrate that Mod($\{\mathfrak{A}\}$) $\setminus$ ALS is not closed under the operation of taking ultraproducts. In order to do that we construct a sequence $\{\mathcal{L}_n\}_{n\in\omega}$ of structures of Mod($\{\mathfrak{A}\}$) $\setminus$ ALS such that $\prod_{n\in\omega} \mathcal{L}_n/\mathcal{U}$ is ALS ($\mathcal{U}$ being non-principal).

Since the explicit definition of the intended $\mathcal{L}_n$ would be very clumsy I think it better to restrict myself to a sort of the geometrical description. Let the geometrical description of the graph representing $\mathfrak{A}_n$ (without $\Box_n$) in fig. 1 be simplified by a box with three inputs I_1, I_2, I_3 and an output O like in fig. 7:

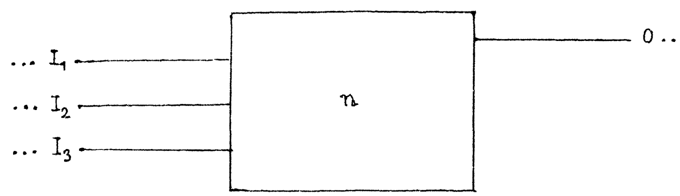

Fig. 7

Imagine the points $\langle x,y\rangle$ with $0 \leq x \leq 2n+3$ lie within the interior of the box while the points of $\{\langle x,y\rangle : x<0 \wedge y = 2\}$, $\{\langle x,y\rangle : x<0 \wedge y \in \{0,1\}$, $\{\langle x,y\rangle : x>2n+3 \wedge y = 3\}$ belong to I_1, I_2, I_3, 0, respectively. Then, by definition of $\mathcal{A}_n$, we can say that "the interior of the box causes the impossibility of actualizing the points of I_1".

Let $\mathcal{S}_n$ be represented by the following collection of boxes $B_1, B_2, \ldots$ connected as described in fig. 8 (B_1 being the "earliest" box; a "latest" box does not exist):

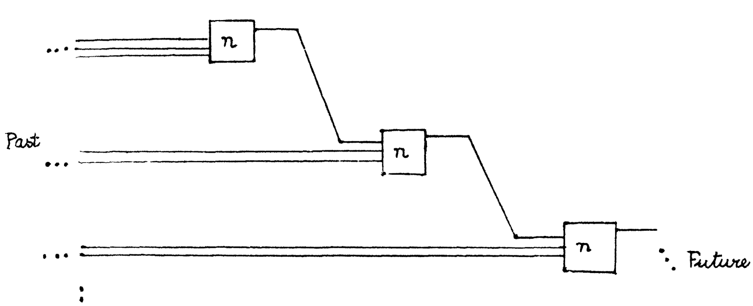

Fig. 8

The construction is not quite uniquely determined since the length of the connections between the boxes are not determined, but every such construction satisfying the following assumption will do.

Assume that the connections between B_n and B_{n+1} are so "long" that, if $\langle x,y \rangle$ and $\langle x',y' \rangle$ belong to different boxes, $|x - x'| > n$. This assumption may be abbreviated by $(\ast)$.

Now let us regard the ultraproduct $\prod_{n \in \omega} \mathcal{L}_n/\mathfrak{U}$. The moments of $\prod_{n \in \omega} \mathcal{L}_n/\mathfrak{U}$ can be coded by couples $\langle \alpha, n \rangle$ in the same way as in $\prod_{n \in \omega} \mathfrak{A}_n/\mathfrak{U}$. The substructure belonging to $\langle \alpha, n \rangle$ is defined analogously. Every such substructure is representable by a combination of graphs like in fig. 2,3,4,5, 6; because of $(\ast)$, any such representation contains at most one of the subgraphs represented by fig. 2,3. By same arguments as in the proof of Theorem 1, we conclude that $\prod_{n \in \omega} \mathcal{L}_n/\mathfrak{U}$ is ALS (even UALS). ∎

Seen from an intuitive point of view, the use of the terminus "possible event" is justified only when dealing with proper cause-and-effect structures (respectively, with universally actualizable law structures). Therefore, the theorems given here may be interpreted as an argument for that the intuitive contents of the notion of possibility cannot be reasonably characterized by a finite number of axioms in a first order language.

Let us regard some decidability questions. Of course, ALS cannot be expected to be decidable. Even subclasses of ALS the structure of which seems rather simple turn out undecidable, in view of the following

Theorem 4: $ALS \cap Mod(\{\forall t\ \exists !!s \Diamond (t,s)\})$ is undecidable.

Proof. Let $R \subseteq \omega \times \omega$ be the QUINE relation (see [3]); then, full arithmetics is interpretable in $\langle \omega, R \rangle$, hence, it remains to show that $\langle \omega, R \rangle$ is interpretable in some model of $ALS \cap Mod(\{\forall t\ \exists !!s \Diamond (t,s)\})$.

Let $\mathcal{L} = \langle (\omega^* + \omega) \cup (\{s_0\} \times \omega), <, +, \Diamond, \mapsto_R \rangle$ where $\langle \omega^* + \omega, <, + \rangle$ is isomorphic to the additive group of integers, and:

$$\Diamond = \{\langle x, \langle s_0, x \rangle\rangle : x \in \omega^* + \omega\}$$

$$\mapsto_R = \{\langle x, \langle s_0, x\rangle, y, \langle s_0, y\rangle\rangle : (x > 0 \wedge y > 0 \wedge Rxy \wedge x < y) \vee (x < 0 \wedge y > 0)\}.$$

Clearly, $\mathcal{L}$ is a structure belonging to $ALS \cap Mod(\{\forall t\ \exists !!s \Diamond (t,s)\})$ in which $\langle \omega, R \rangle$ is interpretable (the interpretability of $\langle \omega, R \rangle$ is based on the symmetry of R). ■

Let $S \subseteq \omega^2$ be an arbitrary infinite relation which is either symmetric, or antisymmetric such that $\forall x,y (Sxy \rightarrow x < y)$. Then, $\langle \omega, S \rangle$ can be shown to be interpretable in some model of $ALS \cap Mod(\{\forall t\ \exists !! \Diamond (t,s)\})$ by the method just used in the proof of Theorem 4. In regard of this, $ALS \cap Mod(\{\forall t\ \exists !!s \Diamond (t,s)\})$ can be shown to be universal ("R-universal" in the sense of [4]) with respect to interpretability.

Finally, decidability of special causality structures is discussed. The causality structures given by examples 1,2 of [2] are interpretable within the real plane and hence decidable.

In case these examples are variated in such a way that there are infinitely many "forking points", interpretability within the real plane will be lost. Therefore, it seems that there are very "few" causality structures the theory of which can be expected to be decidable.

<u>Theorem 5</u>: There is an undecidable $\mathfrak{D} = \langle T \cup S, <, +, \Diamond, \mapsto, \Box \rangle \in$ PCES such that $\mathfrak{D}' = \langle T \cup S, <, +, \Diamond, \mapsto \rangle$ is decidable.

Proof. Let T = R (the real numbers), S = R×R, $\langle R, <, + \rangle$ the additive group of reals, and:

$$\Diamond = \{\langle x, \langle 0, x \rangle\rangle : x \in R\} \cup \{x, \langle 1, x \rangle\} : x \in R \wedge x > 1\}$$

$$\mapsto = \{\langle x, \langle 0, x \rangle, y, \langle 0, y \rangle\rangle : x < y \leq 1\} \cup$$

$$\{\langle x, \langle 0, x \rangle, y, \langle 0, y \rangle\rangle : 1 \leq x \wedge y = x^2\} \cup$$

$$\{\langle x, \langle 1, x \rangle, y, \langle 1, y \rangle\rangle : 1 \leq x \wedge (y = x^2 \vee y = x^4).$$

$\langle T \cup S, <, +, \Diamond, \mapsto \rangle$ is interpretable in the real plane and, hence, decidable.

Let $K \subseteq R$ (regarded not as domains but as fields) a subfield such that

a. K is undecidable

b. For every $a \in K$, $\sqrt{|a|} \in K$.

Note that such a field exists by [5]; another construction was already given in [6], but, as pointed out in [7], needs some modification which will be given in [8].

By b. and the property of being a field we have

i) if $\varepsilon \in \{0,1\}, \langle x, \langle \varepsilon, x \rangle\rangle \mapsto \langle y, \langle \varepsilon, y \rangle\rangle$, $x > 1$ and $x \in K$, then $y \in K$

ii) if $\varepsilon \in \{0,1\}, \langle x, \langle \varepsilon, x \rangle\rangle \mapsto \langle y, \langle \varepsilon, y \rangle\rangle$, $x > 1$ and $x \notin K$, then $y \notin K$.

Hence, by i) and ii), for all $x > 1$, $\varepsilon \in \{0,1\}$ we have:

$\langle x, \langle \varepsilon, x \rangle\rangle \mapsto \langle y, \langle \varepsilon, y \rangle\rangle$ if and only if

either $x \in K$, $y \in K$ or $x \notin K$, $y \notin K$.

Hence, $\mapsto$ works, for elements > 1, separately on K and $R \setminus K$.

Therefore,

$$\Box = \{x, \langle 0, x\rangle : x \leq 1\} \cup$$
$$\{\langle x, \langle 0, x\rangle\rangle : x > 1 \wedge x \in K\} \cup$$
$$\{\langle x, \langle 1, x\rangle\rangle : x > 1 \wedge x \notin K\}$$

represents an actuality relation for $T \cup S, \langle <, +, \Diamond, \mapsto \rangle$, i.e. $\mathfrak{D} = \langle T \cup S, <, +, \Diamond, \mapsto, \Box \rangle \in$ CES (it can easily be seen that $\mathfrak{D} \in$ PCES).

It makes no difficulty to prove that the field K is interpretable in $\mathfrak{D}$. The events of shape $\langle x, \langle 0, x\rangle\rangle$, $x > 1$ can be characterized by $\exists !! \langle t_1, s_1\rangle (\langle t, s\rangle \mapsto \langle t_1, s_1\rangle)$. Let the elements of K greater than 1 correspond to the events of shape $\langle x, \langle 0, x\rangle\rangle \in \Box$, $x > 1$. The addition in K is that of T, and the multiplication in K is definable by using the definability of the relation $f(x) = x^2$ (this definability is based on the choice of $\mapsto$) and $2xy = (x + y)^2 - x^2 - y^2$.

By interpretability of K in $\mathfrak{D}$ and a., $\mathfrak{D}$ is undecidable. ■

Analyzing the last proof and the proofs referred to in it every structure of cardinality $\leq \omega$ can be shown to be interpretable in some actualization of $\mathfrak{D}'$. Thus, PCES $\cap$ Mod(Th($\mathfrak{D}'$)) is even universal with respect to interpretability.

R e f e r e n c e s

[1] G.H. VON WRIGHT: On the logic and epistemology of the Causal relation, Studies in Logic, Methodology and Philosophy of Science IV, Proceedings, North-Holland, 1971.

[2] K. HAUSCHILD: Some remarks concerning the logic of causality, to appear in Wiss. Zeitschr. der Humboldt-Univ., Berlin.

[3] A. CHURCH and Q. QUINE: Some theorems on Definability and Decidability, Journ. Symb. Logic 17(1953),179-187.

[4] K. HAUSCHILD: Über die Universalität axiomatisierbarer Fragmente der Zahlentheorie, Zeitschr. f. math. Logik 18(1972), 255-279.

[5] M. ZIEGLER: Einige unentscheidbare Körpertheorien, Manuskript, Bonn 1979.

[6] G. FICHT: Zur Theorie der pythagoräischen Körper, Diplomarbeit, Konstanz, 1979.

[7] K. HAUSCHILD: Rekursive Unentscheidbarkeit der Theorie der pythagoräischen Körper, Fund. Math 82(1974), 191-197.

[8] K. HAUSCHILD: Interpretierbarkeit in der Körpertheorie I,II (in preparation).

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