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THE INTERIOR REGULARITY AND THE LIOUVILLE PROPERTY
FOR THE QUASILINEAR PARABOLIC SYSTEMS
O. JOHN

Abstract: It is proved that the Liouville property of parabolic quasilinear system - i.e. the fact that each bounded weak solution in R^{n+1} is constant - implies the $C^{0,\alpha}$ -regularity of all bounded weak solutions in arbitrary domain. Similar results for quasilinear elliptic systems were established in [3] - [5].

Key words: Quasilinear parabolic system, interior regularity, parabolic Liouville property.

Classification: 35K55

Denote $z = (t, x) = (t, x_1, \dots, x_n) \in R^{n+1}$ and let $u = (u^1, u^2, \dots, u^m)$ be a vector function. We consider the system

$$(1) \quad \frac{\partial u^i}{\partial t} - \frac{\partial}{\partial x_\alpha} \left(a_{ij}^{\alpha\beta}(u) \frac{\partial u^j}{\partial x_\beta} \right) = 0, \quad i = 1, \dots, m,$$

which we shall write for the sake of brevity as

$$(2) \quad u_t - \operatorname{div}_x(A(u)D_x u) = 0.$$

The coefficients $a_{ij}^{\alpha\beta}$ are supposed to be continuous on R^m and

$$(3) \quad (A(u)\eta, \eta) = a_{ij}^{\alpha\beta}(u) \eta_\alpha^i \eta_\beta^j > 0 \text{ for all } \eta \neq 0, u \in R^m.$$

In what follows we shall write for the vector function $u = \{u^i\}_{i=1}^m$ $u \in L_2(Q)$ instead of $u^i \in (L_2 Q)$, $i = 1, \dots, m$.

Let $Q \subset R^{n+1}$ be a domain, not necessarily bounded. We

say that the function $u \in W_{2,loc}^{0,1}(Q)$ is a weak solution of the system (1) in the domain Q if for each $\varphi \in \mathcal{D}(Q)$ we have

$$(4) \quad \int_Q [u \varphi_t - A(u) D_x D_x \varphi] dz = 0.$$

(The space $W_{2,loc}^{0,1}(Q)$ is the linear set of all functions u such that u^i and $D_x u^i$ are in $L_{2,loc}(Q)$ for all $i = 1, \dots, m$. On each $Q' \subset\subset Q$, Q' bounded, the seminorm

$$\|u\|_{0,1,Q'} = \|u\|_{L_2(Q')} + \|D_x u\|_{L_2(Q')}$$

can be introduced for all $u \in W_{2,loc}^{0,1}(Q)$.)

The system (1) is said to be regular in a domain Q if each weak solution u of (1) in Q which is bounded belongs to $C^{0,\alpha/2,\alpha}(Q)$.

The space $C^{0,\alpha/2,\alpha}(Q)$ is the linear set of all functions continuous on Q for which on each compact $Q' \subset\subset Q$ the expression

$$\sup \left\{ \frac{|u(t,x) - u(t',x')|}{|t - t'|^{\alpha/2} + |x - x'|^\alpha} ; (t,x) \in Q', (t',x') \in Q', (t,x) \neq (t',x') \right\}$$

is finite.

Finally, we say that the system (1) has parabolic Liouville property if for each weak solution u of (1) in the whole R^{n+1} holds the implication

$$(5) \quad \|u\|_{L_\infty(R^{n+1})} < \infty \implies u \text{ is a constant vector function.}$$

Theorem 1. Let the system (1) have parabolic Liouville property. Then it is regular in each domain $Q \subset R^{n+1}$.

Proof. Denote for $R > 0$, $z_0 \in R^{n+1}$

$$(6) \quad Q(z_0, R) = (t_0 - R^2, t_0 + R^2) \times B(x_0, R),$$

where $B(x_0, R)$ is n -dimensional ball in R^n with the radius R and

the center x_0 . Denote further by $u_{z_0, R}$ the integral mean value

$$(6') \quad u_{z_0, R} = \text{mes}^{-1} Q(z_0, R) \int_{Q(z_0, R)} u(z) dz.$$

As it was proved in [1], if for the weak solution u of (1) holds in some point $z_0 \in Q$ that

$$(7) \quad \liminf_{R \rightarrow 0_+} \left[R^{-n-2} \int_{Q(z_0, R)} |u(z) - u_{z_0, R}|^2 dz \right] = 0,$$

then there exists $Q(z_0, \varphi)$ such that $u \in C^{0, \alpha/2, \alpha} Q(z_0, \varphi)$.

(The points for which (7) holds are called the regular points of the weak solution.)

So we want to prove that for each bounded weak solution u of (1) the condition (7) is satisfied in all points $z_0 \in Q$.

Let Q , u and z_0 be fixed, $Q(z_0, R) \subset \subset Q$. Substitute

$$(8) \quad \tau = \frac{t-t_0}{R^2}, \quad \xi = \frac{x-x_0}{R},$$

$$u_R(\tau, \xi) = u(t_0 + R^2\tau, x_0 + R\xi).$$

For an arbitrary constant vector ϕ , we can transform

$$(9) \quad R^{-n-2} \int_{Q(z_0, R)} |u(z) - u_{z_0, R}|^2 dz \leq$$

$$\leq R^{-n-2} \int_{Q(z_0, R)} |u(z) - \phi|^2 dz =$$

$$= \int_{Q(0, 1)} |u_R(\tau, \xi) - \phi|^2 dz d\xi.$$

(In the first inequality we used the fact that the functional $I(\phi) = \int_{Q(z_0, R)} |u(z) - \phi|^2 dz$ attains its minimum in the point $\phi = u_{z_0, R}$.)

It is easy to see from (9) and (7) that z_0 is a regular point of u if one can find a subsequence $\{u_{R_n}\}$ ($R_n \rightarrow 0$) of

$\{u_R\}$ such that

$$(10) \quad u_R \rightarrow p \text{ in } L_2(Q(0,1)),$$

(11) p is a constant vector function.

To prove (10) and (11) we return to the system (1). Substituting into (4) for t , x and u from (8), we obtain that $u_R(\tau, \xi)$ solves the following system:

$$(12) \quad \int_{(Q)_R} [u_R \varphi_\tau - A(u_R) D_\xi u_R D_\xi \varphi] d\tau d\xi = 0.$$

Here $(Q)_R$ is the image of Q in the transformation (8).

For $R \rightarrow 0+$ $(Q)_R$ expands to the whole R^{n+1} , so that if we choose some fixed $K > 0$, then $Q(0,K) \subset \subset (Q)_R$ for all R smaller than some $R(K)$. So, choosing φ with the support lying in $Q(0,K)$, we can see that each u_R solves the system

$$(13) \quad \int_{Q(0,K)} [u_R \varphi_\tau - A(u_R) D_\xi u_R D_\xi \varphi] d\tau d\xi = 0,$$

if only $R < R(K)$.

Writing now in (13)

$$A_R(\tau, \xi) = A(u_R(\tau, \xi)), \quad R < R(K)$$

we can see immediately that we can interpret (13) as a class of the linear parabolic systems with the bounded and measurable coefficients. Because of both the estimate

$$\|u_R\|_{L_\infty(Q(0,K))} \leq \|u\|_{L_\infty(Q)}$$

and the continuity of $A(u)$ we can deduce that the coefficients A_R are equi-bounded and that the corresponding systems have the same constant γ of ellipticity:

$$(A_R(\tau, \xi) \eta, \eta) \geq \gamma |\eta|^2.$$

(The constant γ as well as the upper bound of $|A_R|$ depend only on $\|u\|_{L_\infty(Q)}$.)

Using the lemmas 4 and 5 from [2] we obtain

$$(14) \quad \|u_R\|_{W_2^{1/2,1}(Q(0,K/2))} \leq c \|u_R\|_{L_2(Q(0,K))} \leq$$

$$\leq c^*(K, \|u\|_{L_\infty(Q(0,K))}),$$

where $W_2^{1/2,1}(Q(0,R))$ is a space of all measurable on $Q(0,R)$ functions w for which the expression $\|w\|_{W_2^{1/2,1}(Q(0,R))} =$

$$= \|w\|_{L_2(Q(0,R))} + \|D_x w\|_{L_2(Q(0,R))} + \\ + \int_{B(0,R)} \int_{-R^2}^{R^2} \int_{-R^2}^{R^2} \frac{|u(t,x) - u(x,s)|^2}{(t-s)^2} dt ds dx$$

is finite.

Because of the compactness of the imbedding of $W_2^{1/2,1}$ into L_2 it follows from (14) that we can choose the subsequence $\{u_n\} = \{u_{R_n}\}$ for which

$$u_n \rightarrow p \text{ in } L_2(Q(0,K/2))$$

$$D_x u_n \rightarrow D_x p \text{ in } L_2(Q(0,K/2)),$$

$$u_n \rightarrow p \text{ almost everywhere in } Q(0,K/2).$$

Using the diagonal method (enlarging $Q(0,K/2)$) we reach the subsequence $\{u_n\} = \{u_{R_n}\}$ of $\{u_{R_n}\}$ with the following properties:

$$u_n \rightarrow p \text{ almost everywhere on } R^{n+1},$$

$$(16) \quad u_n \rightarrow p \text{ in each } L_2(\Omega), \Omega \text{ is bounded in } R^{n+1},$$

$$D_x u_n \rightarrow D_p \text{ in each } L_2(\Omega), \Omega \text{ is bounded in } R^{n+1}.$$

From here it follows - after passing to the limit in (12) - that p is a weak solution of (1) in R^{n+1} , so that p is a constant vector function because of Liouville parabolic property.

From (9) we get, putting $\phi = p$ and $R = R_n$, that

$$\lim_{n \rightarrow \infty} R^{-n-2} \int_{Q(z_0, R_n)} |u(z) - u_{z_0, R_n}|^2 dz = 0.$$

From here it follows immediately (7), q.e.d.

R e f e r e n c e s

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