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Jiří Vinárek

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REPRESENTATIONS OF COMMUTATIVE SEMIGROUPS BY PRODUCTS OF METRIC O-DIMENSIONAL SPACES

Jiří VINÁREK

Abstract: For every commutative semigroup $(S,+)$ there is constructed a collection $\{r(s); s \in S\}$ of complete metric 0-dimensional spaces such that the following conditions hold:
(i) $r(s + s')$ is isometric to $r(s) \times r(s')$,
(ii) $r(s)$ is homeomorphic to $r(s')$ iff $s = s'$

Key words: Semigroup, representation, product, 0-dimensional space.

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Secondary 20M30

Isomorphisms of products have been studied for various algebraic, relational and topological structures One of original problems was to find a topological space X which is homeomorphic to X^3 but not to X^2. After solving this problem, this question was investigated in special categories. A construction of an object X which is isomorphic to X^3 but not to X^2 is a special case of a representation of a commutative semigroup by products in a category, investigated by V. Trnková and the participants of the Seminar on General Mathematical Structures in Prague. A survey of this topic is given in [4]. Nevertheless, let us recall Trnková s result ([5]) that every compact metric 0-dimensional space X which is homeomorphic to X^3 is also homeomorphic to X^2.

The aim of this paper is to prove the following:

Theorem. For any commutative semigroup $(S,+)$ there exists a collection $\{r(s); s \in S\}$ of complete metric 0-dimensional spaces such that the following conditions hold:

(i) $r(s + s')$ is isometric to $r(s) \times r(s')$

(ii) $r(s)$ is homeomorphic to $r(s')$ iff $s = s'$

Remarks. 1. As a special case of Theorem we obtain a complete metric 0-dimensional space X isometric to X^3 but not homeomorphic to X^2.

2. The theorem strengthens the Trnková's result 3. from [3]: the same theorem is proved in [3], except the fact that the spaces $r(s)$ are 0-dimensional. Nevertheless, the construction of 0-dimensional spaces $r(s)$ requires more subtle argumentation.

I am indebted to V. Trnková for valuable suggestions and reading the manuscript.

1. Conventions and notations. We shall use the symbol $\sim$ for a homeomorphism, $\cong$ for an isometry of spaces. Since the construction needs also metrizability of infinite products, our basic category $\underline{C}$ will be that of complete metric spaces with a diameter ≤ 1 and contractions (i.e. Lipschitz mappings with a Lipschitz constant ≤ 1). This category has all products (denoted by $\prod$, or x for finite collections) and all coproducts (denoted by $\amalg$). Actually, if I is a set and $\{(X_\iota, \rho_\iota); \iota \in I\}$ is a collection of objects of $\underline{C}$ then $\prod_{\iota \in I} (X_\iota, \rho_\iota) = (\prod_{\iota \in I} X_\iota, \rho)$ where $\rho((x_\iota)_{\iota \in I}, (y_\iota)_{\iota \in I}) = \sup_{\iota \in I} \rho_\iota(x_\iota, y_\iota)$. Moreover, one can see easily that the functor $\mathcal{F}: \underline{C} \to \underline{TOP}$ assigning to each metric

space (X,ρ) a topological space with the topology induced by ρ, preserves finite products (and all coproducts).

2. Denote by N the additive semigroup of non-negative integers and by N^{α} its α-th power, i.e. the semigroup of all the functions on α with values in N, where the operation + is defined point-wise. exp N is the semigroup of its subsets with + defined by

$$A + A' = \{a + a';\, a \in A,\ a' \in A'\}.$$

Denote by N^{+} the set of all the positive integers.

By [4], any commutative semigroup S is isomorphic to a subsemigroup of exp $N^{\aleph_0 \cdot \mathrm{card}\, S}$. Hence, for a representation of any commutative semigroup by products of complete metric 0-dimensional spaces, it is sufficient to construct for any subset A of $N^{\aleph_0 \cdot \mathrm{card}\, S}$ a complete metric 0-dimensional space X(A) such that the following two conditions hold:

(i) $X(A + A') \cong X(A) \times X(A')$

(ii) $X(A) \sim X(A')$ iff $A = A'$

Since the distributivity of finite products of objects of $\underline{C}$ is fulfilled, it suffices - due to Trnková s result ([4]) - to construct for any function $f \in N^{\aleph_0 \cdot \mathrm{card}\, S}$ a complete metric 0-dimensional space X(f) with a diameter ≤ 1 such that for every f, $g \in N^{\aleph_0 \cdot \mathrm{card}\, S}$ and $A, A' \subseteq N^{\aleph_0 \cdot \mathrm{card}\, S}$ the following conditions hold:

(1) $X(f + g) \cong X(f) \times X(g)$

(2) $\coprod_{2^{\aleph_0 \cdot \mathrm{card}\, S}} \left(\coprod_{h \in A} X(h) \right)$ is 0-dimensional

(3) $\coprod_{2^{\aleph_0 \cdot \mathrm{card}\, S}} \left(\coprod_{h \in A} X(h) \right) \sim \coprod_{2^{\aleph_0 \cdot \mathrm{card}\, S}} \left(\coprod_{k \in A'} X(k) \right)$

iff $A = A'$

where $\coprod_{2^{\alpha}} Z$ denotes the coproduct of 2^{α} copies of Z.

(Having constructed $X(f)'$s satisfying (1)-(3) one can put $X(A) = \coprod_{2^{\aleph_0}\cdot \operatorname{card} S} \left(\coprod_{f \in A} X(f) \right)$. Clearly, conditions (i) and (ii) are satisfied.)

Trnková's general method for constructing such $X(f)'$s is the following: find a collection $\{X_a ; a \in \aleph_0 . \operatorname{card} S\}$ of objects of a given category such that for every $A, A' \subseteq N^{\aleph_0 \cdot \operatorname{card} S}$ the following condition holds:

$$(*) \quad \coprod_{2^{\aleph_0}\cdot \operatorname{card} S} \left(\coprod_{h \in A} \prod_{a \in \aleph_0 \cdot \operatorname{card} S} X_a^{h(a)} \right) \sim$$

$$\sim \coprod_{2^{\aleph_0}\cdot \operatorname{card} S} \left(\coprod_{k \in A'} \prod_{a \in \aleph_0 \cdot \operatorname{card} S} X_a^{k(a)} \right) \text{ iff } A = A'.$$

Then one can define $X(f) = \prod_{a \in \aleph_0 \cdot \operatorname{card} S} X_a^{f(a)}$ and easily check (1) and (3). Since arbitrary coproducts of 0-dimensional spaces in $\underline{C}$ are 0-dimensional, but products of 0-dimensional spaces need not have this property, it will be necessary to prove 0-dimensionality of spaces $X(f)$, too.

3. Construction. Let $\underline{Cn}$ be the class of cardinal numbers. Denote by γ the first ordinal with card $\gamma = \aleph_0 . \operatorname{card} S$. For every $a \in \gamma$ choose a set $B_a = \{\beta_{a,n} ; n \in N^+\} \subseteq \underline{Cn}$ such that the following conditions hold:

$$2^{\gamma} < \beta_{0,1} ; \ \beta_{a,n} < \beta_{a,n+1} ; \ \beta_{a,1} > (\sup \{\beta_b ; b < a\})^{\gamma}$$

where $\beta_b = \sup \{\beta_{b,n} ; n \in N^+\}$.

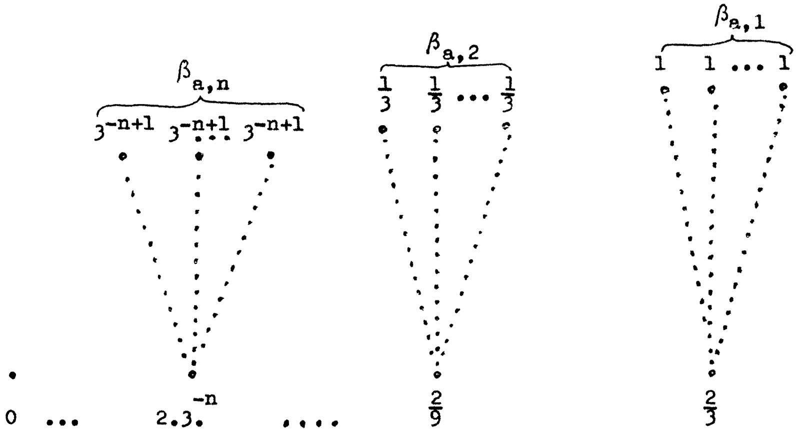
Denote

$B = \bigcup_{a \in \gamma} B_a$. Let $C = [\![0,1]\!] \setminus \bigcup_{n=1}^{+\infty} \bigcup_{i=1}^{\frac{3^n-1}{2}}]\!] \frac{2i-1}{3^n}, \frac{2i}{3^n} [\![$

be the Cantor set (with the usual real-line metric),

$C_n = [\![2.3^{-n}, 3^{-n+1}]\!] \cap C$, $D = \{2.3^{-n} ; n \in N^+\} \cup \{0\}$ (again with the usual metric.

For every $a \in \gamma$ define a metric space X_a by glueing $\beta_{a,n}$ copies of C_n to the point 2.3^{-n} of D (as shown in the picture).



More precisely, $X_a = (\overline{X_a}, \rho_a)$ where

$\overline{X_a} = \bigcup_{n \in N^+}((C_n \setminus \{2.3^{-n}\}) \times \beta_{a,n}) \cup D$,

$\rho_a(x,y) = |x - y|$ whenever $x,y \in D$,

$$\rho_a((x,\alpha),(y,\beta)) = \begin{cases} |x - y| \text{ if } x,y \in C_n \text{ and } \alpha = \beta \\ |x - 2.3^{-n}| + |2.3^{-n} - 2.3^{-m}| + \\ \quad + |y - 2.3^{-m}| \\ \text{if } x \in C_n,\ y \in C_m \text{ and } n \neq m \text{ or } \alpha \neq \beta, \end{cases}$$

$\rho_a((x,\alpha),y) = |x - 2.3^{-n}| + |y - 2.3^{-n}|$ if $x \in C_n$ and $y \in D$.

Denote $\| \ \| : X_a \longrightarrow C$ by $\|x\| = x$ whenever $x \in D$, $\|(y,\alpha)\| = y$ whenever $(y,\alpha) \in X_a \setminus D$.

One can check easily that every X_a is a complete metric 0-dimensional space with diam $X_a = 1$. It remains to prove $(*)$ and 0-dimensionality of $X(f) = \prod_{a \in \gamma} X_a^{f(a)}$ for every $f \in N^{\gamma}$.

4. Recall the definition of a <u>dispersive character</u> (cf. [2]): Let y be a point of a topological space; then a dispersive character $\Delta(y) = \min\{\text{card } V;\ V$ is an open neighbourhood of $y\}$.

Using dispersive characters we can introduce the following:

5. <u>Definition</u>. Let x be a point of a topological space. Then a <u>dispersive type</u> $\tau(x) = \bigcap\{\{\Delta(y); y \in U\};\ U$ an open neighbourhood of $x\}$.

6. <u>Observation</u>. If X, Y are topological spaces, $x \in X$, $y \in Y$, then $\Delta((x,y))$ (in $X \times Y$) is equal to the product of $\Delta(x)$ (in X) and $\Delta(y)$ (in Y).

7. For any $f: \gamma \longrightarrow N$ denote by $L(f)$ the set $\{(a,i);\ a \in \gamma,\ i \in \{1,\dots,f(a)\}\}$. By the associativity of products there is $X(f) = \prod_{a \in \gamma} X_a^{f(a)} = \prod_{(a,i) \in L(f)} X_a$. For any $(a,i) \in L(f)$ denote by $\pi_{a,i}$ the corresponding projection of $X(f)$ onto X_a.

8. <u>Lemma</u>. Let $x \in X(f)$ be given such that there exist $\delta > 0$ with the following property: $|\,\|\pi_{a,i}(x)\| - 2.3^{-n}| \geq \delta$ for any $(a,i) \in L(f)$, $n \in N^+$.
Then $\Delta(x) = (2^{\aleph_0})^{\text{card } A_f}$ where $A_f = \{a; f(a) \neq 0\}$.

<u>Proof</u>. Any non-empty open set in X_a has cardinality at least $2^{\aleph_0}$. Hence, $\Delta(x) \geq (2^{\aleph_0})^{\text{card } A_f}$. On the other hand, $\text{card}\{y \in \overline{X_a}: \rho_a(\pi_{a,i}(x),y) < \delta\} = 2^{\aleph_0}$ for any $a \in A$ and $i \in \{1,\dots,f(a)\}$ and $\text{card}\{y \in X(f); \rho(x,y) < \delta\} = (2^{\aleph_0})^{\text{card } A_f}$.

Q.E.D.

9. <u>Lemma</u>. Let $a \in \gamma$ and $g \in N^{\gamma}$ be given such that

$g(a') = 0$ for any $a' \geq a$. If $x \in X(g)$ then $\Delta(x) \notin B_a$.

Proof. By the construction, $\operatorname{card} X_b = \beta_b$ for any $b \in \gamma$. Hence, $\operatorname{card} X(g) \leq \prod_{b<a} \beta_b < \beta_{a,1}$, $\Delta(x) < \beta_{a,1}$, and therefore $\Delta(x) \notin B_a$. Q.E.D.

10. Lemma. Let $a \in \gamma$ and $h \in N^{\gamma}$ be given such that $h(a') = 0$ for any $a' \leq a$, $x \in X(h)$. Then $\Delta(x) \notin B_a$.

Proof. Let V be an open neighbourhood of x, $b > a$, $i \in \{1, \ldots, h(b)\}$. Consider two cases:

(i) $\pi_{b,i}(V) \cap D = \emptyset$.

Then $\operatorname{card} \pi_{b,i}(V) = 2^{\aleph_0}$.

(ii) $\pi_{b,i}(V) \cap D \neq \emptyset$.

Then $\pi_{b,i}(V)$ contains a neighbourhood W of a point $2.3^{-n} \in X_b$ for a suitable n. Hence, $\operatorname{card} \pi_{b,i}(V) \geq \operatorname{card} W \geq \beta_{b,n} > \beta_a$. Obviously $\operatorname{card} V = \prod_{b \in \gamma} \prod_{i=1}^{h(b)} \operatorname{card} \pi_{b,i}(V)$ and either $\operatorname{card} V = (2^{\aleph_0})^{\operatorname{card} A_h} < \beta_{a,1}$, or $\operatorname{card} V > \beta_a$. Therefore, either $\operatorname{card} V > \beta_a$ for any open neighbourhood V of y and $\Delta(x) > \beta_a$, or $\operatorname{card} V_0 = 2^{\gamma}$ for some neighbourhood V_0 and $\Delta(x) < \beta_{a,1}$. In both these cases $\Delta(x) \notin B_a$. Q.E.D.

11. Lemma. Let $f \in A$, $a \in \gamma$, $n \in N^+$, $x \in X(f)$. Then $\beta_{a,n} \in \tau(x) \Longleftrightarrow \exists\, j \in \{1, \ldots, f(a)\}$ such that $\pi_{a,j}(x) = 2.3^{-n}$.

Proof. A. Suppose that $\pi_{a,j}(x) = 2.3^{-n}$. Let V be an arbitrary open neighbourhood of x; choose a positive integer p such that $\{z;\ \rho(z,x) \leq 3^{-p}\} \subseteq V$. Define $v \in X(f)$ by the following formulas: $\pi_{a,j}(v) = 2.3^{-n}$ and for $(b,i) \neq (a,j)$ there is: $\pi_{b,i}(v) = \pi_{b,i}(x)$ if $\rho_p(\pi_{b,i}(x), D) \geq 3^{-p}$; $\pi_{b,i}(v) = (3^{-p}, 0)$ if $\|\pi_{b,i}(x)\| \leq 3^{-p}$; $\pi_{b,i}(v) = (r + 3^{-p}, \alpha)$ if $\|\pi_{b,i}(x)\| > 3^{-p}$

and $0 < \rho_b(\pi_{b,i}(x),D) < 3^{-p}$ where $\pi_{b,i}(x) = (\| \pi_{b,i}(x) \|, \alpha)$, $r = \max (D \cap [\![0, \| \pi_{b,i}(x) \|]\!])$; $\pi_{b,i}(v) = (r + 3^{-p}, 0)$ if $\pi_{b,i}(x) = r \in D$, $r > 3^{-p}$.

Obviously, $\rho_b(\pi_{b,i}(v),D) \geq 3^{-p-1}$ for any $(b,i) \neq (a,j)$ and $\rho(v,x) \leq 3^{-p}$ (hence, $v \in V$). Denote $A_f' = A_f$ if $f(a) > 1$, $A_f' = A_f \setminus \{a\}$ if $f(a) = 1$. By 6 and 8, $\triangle(v) = (2^{\aleph_0})^{\operatorname{card} A_f'} \circ \beta_{a,n} = \beta_{a,n}$. Hence, $\beta_{a,n} \in \tau(x)$.

B. Suppose that $\pi_{a,i}(x) \neq 2.3^{-n}$ for any $i \in \{1,\dots,f(a)\}$. Denote $M' = \{i;\ \pi_{a,i}(x) \in D \setminus \{0\}\}$, $M'' = \{i;\ \pi_{a,i}(x) = 0\}$, $M = \{1,\dots,f(a)\} \setminus (M' \cup M'')$, $\varepsilon = \min (\{\frac{1}{2}\ \pi_{a,i}(x); i \in M'\} \cup \cup \{\rho_a(\pi_{a,i}(x),D);\ i \in M\} \cup \{3^{-n}\})$, $U = \{z;\ \rho(x,z) < \varepsilon\}$. Let $y \in \in U$ be an arbitrary point; denote $y_1 = (\pi_{b,i}(y))_{b<a, 1 \leq i \leq f(b)}$, $y_2 = (\pi_{a,i}(y))_{i \in M'}$, $y_3 = (\pi_{a,i}(y))_{i \in M''}$, $y_4 = (\pi_{a,i}(y))_{i \in M}$, $y_5 = (\pi_{b,i}(y))_{b>a, 1 \leq i \leq f(b)}$.

By Lemmas 9 and 10, $\triangle(y_1) \neq \beta_{a,n}$, $\triangle(y_5) \neq \beta_{a,n}$. Obviously, $\triangle(y_4) = (2^{\aleph_0})^{\operatorname{card} M} \neq \beta_{a,n}$.

If $i \in M'$ then either $\pi_{a,i}(y) = \pi_{a,i}(x) = 2.3^{-m}$ (where $m \neq n$) and $\triangle(\pi_{a,i}(y)) = \beta_{a,m}$, or $\pi_{a,i}(y) \notin D$ and $\triangle(\pi_{a,i}(y)) = 2^{\aleph_0}$. Observation 6 implies that $\triangle(y_2) = \max \{\triangle(\pi_{a,i}(y);\ i \in \in M'\} \neq \beta_{a,n}$.

For $i \in M''$ one must consider three cases:

(i) $\pi_{a,i}(y) = 0$

(ii) $\pi_{a,i}(y) = 2.3^{-m}$

(iii) $\pi_{a,i}(y) \notin D$

In the case (i) there is $\triangle(\pi_{a,i}(y)) = \beta_a \neq \beta_{a,n}$; in the case (ii) there is $\triangle(\pi_{a,i}(y)) = \beta_{a,m} \neq \beta_{a,n}$ (since $\rho(x,y) < 3^{-n}$ and $\pi_{a,i}(x) = 0$, there is $m > n$); in the case (iii) there is $\triangle(\pi_{a,i}(y)) = 2^{\aleph_0}$. Consequently, one obtains by Observation

6 that $\Delta(y_3) \neq \beta_{a,n}$. According to 6, $\Delta(y) = \Delta(y_1) \cdot \Delta(y_2) \cdot \Delta(y_3) \cdot \Delta(y_4) \cdot \Delta(y_5) \neq \beta_{a,n}$. Hence, $\beta_{a,n} \notin \tau(x)$. Q.E.D.

12. Denote $\widetilde{X(A)} = \{x \in X(A);\ \tau(x) \cap B = \emptyset\}$.

Now, we can prove the following:

13. <u>Lemma</u>. If $f \in A$, $x \in X(f)$ then $x \in \widetilde{X(A)}$ iff for every $a \in \gamma$ and every $i \in \{1,\dots,f(a)\}$: $\pi_{y,i}(x)$ is not in $D \setminus \{0\}$.

<u>Proof</u> follows from Lemma 11.

14. For every open $U \neq \emptyset$ define $F(U): \gamma \longrightarrow N$ by $F(U)(a) = \sup\{\text{card}\ (\tau(y) \cap B_a);\ y \in U\}$.

Then for every $x \in \widetilde{X(A)}$ define $F(x): \gamma \longrightarrow N$ by $F(x)(a) = \min\{F(U)(a);\ U$ an open neighbourhood of $x\}$.

15. <u>Lemma</u>. $F(x)(a) = \text{card}\{i;\ \pi_{a,i}(x) = 0\}$ for every $x \in \widetilde{X(A)}$.

<u>Proof</u>. Denote $J = \{i;\ \pi_{a,i}(x) = 0\}$, card $J = k$.

a) Let U be an open neighbourhood of x, $y \in U$ such that for any $j \in J$ there is $\pi_{a,j}(y) \in D \setminus \{0\}$ with $j \neq j' \Longrightarrow \pi_{a,j}(y) \neq \pi_{a,j'}(y)$ and $\pi_{a,j}(y) \notin D$ for any $j \notin J$. By Lemma 11, card $(\tau(y) \cap B_a) = k$ and $F(U)(a) \geq k$. Therefore, $F(x)(a) \geq k$.

b) On the other hand, denote $\varepsilon = \min\{\rho_a(\pi_{a,j}(x), D);\ j \in \{1,\dots,f(a)\} \setminus J\}$. Let U be an open neighbourhood of x such that $U \subseteq \{z;\ \rho(z,x) < \varepsilon\}$, $y \in U$. Clearly, $\pi_{a,j}(y) \notin D$ for every $j \in \{1,\dots,f(a)\} \setminus J$. By Lemma 11, card $(\tau(y) \cap B_a) \leq$ $\leq$ card $(\{\pi_{a,i}(y);\ i = 1,\dots,f(a)\} \cap (D \setminus \{0\})) \leq k$. Hence, $F(U)(a) \leq k$ for arbitrary sufficiently small U and $F(x)(a) \leq k$, too.

16. <u>Lemma</u>. If $x \in X(f) \cap \widetilde{X(A)}$ such that, for every $a \in \gamma$

and every $1 \leq i \leq f(a)$: $\pi_{a,i}(x)$ is equal to 0, then $F(x) = f$.

Proof follows firectly from Lemma 15.

17. Define $X(A)_{max} = \{x \in \widetilde{X(A)};\ \exists U$ an open neighbourhood of x such that for every $y \in \widetilde{X(A)} \cap (U \setminus \{x\})$ there exists $a \in \gamma$ such that $F(y)(a) < F(x)(a)\}$.

18. Lemma. $X(A)_{max} = \{x \in \widetilde{X(A)};\ \pi_{a,i}(x) = 0$ for every $(a,i)\}$.

Proof. a) If $\pi_{a,i}(x) = 0$ for every (a,i) then for $U = \{z;\ \rho(x,z) < 1\}$ and $y \in U \setminus \{x\}$ there exists a couple (a,i) such that $\pi_{a,i}(x) \neq 0$. By Lemma 15, $F(y)(a) < F(x)(a)$. Hence, $x \in X(A)_{max}$.

b) Suppose that there exists a couple (a,i) such that $\pi_{a,i}(x) \neq 0$. Since $x \in \widetilde{X(A)}$, according to Lemma 13 $\pi_{a,i}(x) \notin D$ and $\pi_{a,i}(x) = (u,\alpha)$ with $u \in C \setminus D$. Since C has no isolated point, for any open neighbourhood U of x there exists $y \in U \setminus \{x\}$ such that $\pi_{a,i}(y) \notin D$ and for any $(a',i') \neq (a,i)$ there is $\pi_{a',i'}(y) = \pi_{a',i'}(x)$.
One can see easily that $y \in \widetilde{X(A)}$ and $F(y) = F(x)$. Hence, $x \notin X(A)_{max}$. Q.E.D.

19. Proposition. $A = \{F(x); x \in X(A)_{max}\}$.

Proof follows from Lemmas 16 and 18.

20. Corollary. If $A \neq A'$ then $F(A) \not\sim F(A')$.

Proof follows directly from Proposition 19.

21. Before proving 0-dimensionality of $X(A)$ recall the following:

Lemma. For any point $c \in C$ such that $3^n c \in N$ the set

$\{d \in C;\ |d - c| \leq 3^{-n-1}\}$ is equal to $\{d \in C;\ |d - c| < 2.3^{-n-1}\}$.

Proof. The construction of the Cantor set C implies that $3^n c \in N \Rightarrow\ ]c + 3^{-n-1}, c + 2.3^{-n-1}[\ \cap C = \emptyset$, $]c - 2.3^{-n-1}, c - 3^{-n-1}[\ \cap C = \emptyset$. Hence, $\{d \in C;\ |d - c| < 2.3^{-n-1}\} = \{d \in C;\ |d - c| \leq 3^{-n-1}\}$. Q.E.D.

22. Proposition. X(A) is a 0-dimensional space.

Proof. It suffices to prove that there exists a σ-locally finite clopen basis. For every $n \in N$ put $P_n = \{x \in X(A);\ 3^n \| \pi_{a,i}(x) \| \in N$ for any $(a,i)\}$, $\mathcal{B}_n = \{\{y; \rho(y,x) \leq 3^{-n-1}\};\ x \in P_n\}$.

If x, z are distinct points of P_n then $\rho(x,z) \geq 3^{-n} > 2.3^{-n-1}$. Hence, $\mathcal{B}_n$ is a discrete system. Lemma 21 implies that any element of $\mathcal{B}_n$ is clopen.

Let U be open in $X(f) \subseteq X(A)$, $z \in U$, $n \in N$ such that $\{y: \rho(y,z) < 3^{-n}\} \subseteq U$. For any $a \in \gamma$, $1 \leq i \leq f(a)$ define $x_{a,i} \in P_n$ such that $\rho_a(x_{a,i}, \pi_{a,i}(z)) \leq 3^{-n-1}$ ($3^n \| x_{a,i} \|$ is the closest integer to $3^n \| \pi_{a,i}(z) \|$). Denote by x the point of X(f) with $\pi_{a,i}(x) = x_{a,i}$ for any $a \in \gamma$, $1 \leq i \leq f(a)$, $V_z = \{y; \rho(y,x) \leq 3^{-n-1}\} \in \mathcal{B}_n$. Obviously, $\{z\} \subseteq V_z \subseteq \{y; \rho(y,z) < 3^{-n}\} \subseteq U$ and $\bigcup_{z \in U} V_z = U$.

Therefore, $\mathcal{B} = \bigcup_{n \in N} \mathcal{B}_n$ is a σ-discrete clopen basis and X(A) is 0-dimensional. Q.E.D.

23. Corollary 20 and Proposition 22 finish the proof of Theorem.

24. Remark. In [1], sum-productive representations of ordered commutative semigroups are investigated. The above construction and results of [1] give immediately the following result:

For every ordered commutative semigroup $(S,+,\leq)$ there exists a collection $\{r(s); s \in S\}$ of complete metric 0-dimensional spaces such that the following conditions hold:

(i) $r(s + s')$ is isometric to $r(s)$ $r(s')$;

(ii) $r(s)$ is homeomorphic to $r(s')$ iff $s = s'$;

(iii) $r(s)$ is homeomorphic to a clopen subset of $r(s')$ iff $r(s)$ is isometric to a clopen subset of $r(s')$, and this is fulfilled iff $s \leq s'$.

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Matematický ústav, Univerzita Karlova, Sokolovská 83, 18600 Praha 8, Czechoslovakia

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