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PRODUCTS OF CONVERGENCE PROPERTIES

J. GERLITS and Zs. NAGY

Abstract: The impetus for this paper was given by P. Simon's recent beautiful example for two compact Fréchet spaces whose product is not Fréchet. Our aim is to study the behaviour of convergence properties with respect to the products. The main new results are: the product of two w-spaces, one of which is compact, is a w-space; the product of a compact Fréchet space and of a w-space is a Fréchet space. We give also an example for two w-spaces whose product has not even tightness.

Key words: Convergence properties, topological product, compact spaces.

Classification: 54D50, 54D55, 54B10

In this paper by a convergence property we shall mean one of the following properties:

(1) X is first countable.

(2) X is a W -space [4]: for each $p\in X$ the first player has a winning strategy in the Gruenhage's game on $\langle X,p\rangle$

(3) X is a G -space (G stands for Gruenhage) i.e. $t(X)=\omega$ and each countable subset of X is first countable.

(4) X is a w -space: for each $p\in X$ the second player does not have a winning strategy in the Gruenhage's

game on $\langle X,p\rangle$; equivalently [7], if $A_n \subset X$, $p \in \bar{A}_n$ ($n<\omega$) then there is a sequence $p_n \in A_n$ with $\lim p_n = p$.

(5) X is Fréchet: if $p \in X$, $A \subset X$ and $p \in \bar{A}$ then there is a sequence $p_n \in A$ with $\lim p_n = p$.

(6) X is sequential: if $A \subset X$ is not closed then there is a convergent sequence $\{p_n : n<\omega\} \subset A$ with $\lim p_n \notin A$.

(7) X is a k -space: if $A \subset X$ is not closed then there is a compact $C \subset X$ with $C \cap A$ not closed.

(8) $t(X)=\omega$, the tightness of X is countable: if $p \in X$, $A \subset X$ and $p \in \bar{A}$ then there is a countable subset $B \subset A$ with $p \in \bar{B}$.

See [2], [3] and [4] for the proofs of the implications 1)⇒2)⇒3)⇒4)⇒5)⇒6)⇒7), 6)⇒8) and that neither of these implications is reversible. The only open question here is

Problem 1 Is there an example (in ZFC) of a w -space which is not a G -space?

A moment's reflexion shows that if there is such an example then there is also a countable one with only one non-isolated point. Let us note also that assuming Martin's Axiom and the negation of the Continuum Hypothesis, any countable space with weight less than continuum does the job. A more complicated example can be constructed also if the Continuum Hypothesis holds.

Let now $\mathcal{P}$ be any fixed convergence property. Our aim is to characterize those convergence properties $\mathcal{Q}$ for which the product of any $\mathcal{P}$ -space and of any $\mathcal{Q}$ -space has $\mathcal{P}$.

It is immediate that any of the convergence properties is hereditary with respect to closed subsets hence if a product has a certain property then each factor space also does. Consequently the best case is when the product of two $\mathcal{P}$ -spaces is again $\mathcal{P}$ while the worst case is when there is a $\mathcal{P}$ -space and a first countable space whose product is not a $\mathcal{P}$ -space.

The general case

ad 1) The product of two first countable spaces is evidently again first countable.

ad 2) By a theorem of Gruenhage [4] the product of two W -spaces is again a W -space.

ad 3) See Gruenhage's paper [4] for the proof that the product of a G -space and of a W -space is a G -space.

On the other hand, the following example shows that the product of two G -spaces need not be a G -space. This example solves a problem posed by Gruenhage in [4].

Example 1 Two G -spaces whose product does not have tightness ω .

We begin with some set-theoretical definitions.

Let C denote the set of limit ordinals in ω_1 , $C' = C\cup\{0\}$. Two sets A , B are *almost equal* $(A =^* B)$ if their symmetrical difference is finite $(|A \circ B| < \omega)$. An ω_1 *-sequence* is a sequence $\{A_\xi : \xi \in C\}$ having the following properties:

a.) $A_\xi \subset \xi$ for each $\xi \in C$,

b.) if $\xi, \eta \in C$, $\xi < \eta$ then $A_\xi =^* A_\eta \cap \xi$.

An ω_1 -sequence is *trivial* if there is a set $A \subset \omega_1$ such that $A_\xi =^* A \cap \xi$ for each $\xi \in C$.

Assume now that $\{A_\xi:\xi\in C\}$ is a non trivial ω_1 -sequence; using it we construct two G -spaces X, Y with $t(X\times Y)=\omega_1$.

Let the ground sets of X and Y be $\omega_1\cup\{p\}$, $p\notin\omega_1$. The points of ω_1 are isolated in both spaces. For a set $T\subset\omega_1$, $p\in\overline{T}^X$ iff $T\cap A_\xi$ is infinite for a suitable $\xi\in C$. Similarly, $p\in\overline{T}^Y$ if $T\cap(\xi-A_\xi)$ is infinite for some $\xi\in C$. It is immediate that X and Y are G -spaces: any countable subset is the union of a clopen discrete set and of a convergent sequence. On the other hand, $t(X\times Y)=\omega_1$. We must prove that for each $\eta\in C$ η is the union of an X -closed and of a Y -closed set but ω_1 is not such a union because then the point (p,p) is in the closure of the set $S=\{(\xi,\xi):\ \xi<\omega_1\}$ in $X\times Y$ but each countable subset of S is closed.

If $\eta\in C$ then $\eta=(\eta-A_\eta)\cup A_\eta$; $\eta-A_\eta$ is X -closed and A_η is Y -closed. If $A\subset\omega_1$ is Y -closed then for each $\eta\in C$ $A\cap(\eta-A_\eta)$ is finite. Using now that $\{A_\xi:\xi\in C\}$ is not trivial we get an $\eta\in C$ with $A_\eta-A$ infinite but then ω_1-A is not X -closed.

Hence our task is reduced to construct a non-trivial ω_1 -sequence. The following lemma will be helpful.

<u>Lemma</u> If the ω_1 -sequence $\{A_\xi:\xi\in C\}$ is trivial then there are two ordinals $\xi,\eta\in C$, $\xi<\eta$ such that $A_\xi=A_\eta\cap\xi$.

<u>Proof</u> Choose a set $A\subset\omega_1$ such that $A_\xi=^*A\cap\xi$ for each $\xi\in C$ and take the function

$$h(\xi) = (A\cap\xi)\circ A_\xi \quad .$$

Evidently $h(\xi)$ is a finite subset of ξ for $\xi\in C$. Using Neumer's theorem [5] we get $\xi,\eta\in C$, $\xi<\eta$ with $h(\xi)=h(\eta)$. It is immediate now that $A_\xi=A_\eta\cap\xi$. □

Consequently we have to construct a sequence $\{A_\xi:\xi\in C\}$ with the following properties: for $\xi,\eta\in C$, $\xi<\eta$

a.) $A_\xi\subset\xi$,

b.) $0\neq|A_\xi\circ(A_\eta\cap\xi)|<\omega$.

The construction is by transfinite recursion.

We shall also assume that

c.) The system $\mathfrak{A}^\eta_\xi=\{A\subset\eta-\xi:\ \forall\tau\in((\eta+1)-\xi)\cap C$ $0<|(A\circ A_\tau)\cap(\tau-\xi)|<\omega\}$ is infinite for any $\xi,\eta\in C'$, $\xi<\eta$.

Let $A_\omega\subset\omega$ be arbitrary. If $\alpha\in C$, $\{A_\xi:\xi\in C\cap\alpha\}$ is constructed and fulfils a.), b.) and c.) for $\eta<\alpha$, we distinguish two cases.

1.) $\alpha=\beta+\omega$. Let A_α be any member of $\mathfrak{A}^\beta_0$.

2.) $\alpha=\lim\alpha_n$, $\alpha_n\in C$, $\alpha_n<\alpha_{n+1}$ $(n<\omega)$. Choose a set $A_n\in\mathfrak{A}^{\alpha_n}_{\alpha_{n-1}}$ $(\alpha_{-1}=0)$ and put $A_\alpha=\cup\{A_n:n<\omega\}$.

It is now easily seen that a.), b.) and c.) hold also for $\eta=\alpha$. □

Problem 2 Let X and Y be G -spaces. Is it possible that $t(X\times Y)>2^\omega$?

ad 4) By a theorem of Gruenhage [4] the product of a w -space and a W -space is a w -space. On the other hand, Example 1 shows that a G -space is not enough.

ad 5), 6), 7) These properties behave very badly with respect to products as the following example shows.

Example 2 A Fréchet space and a first countable space whose product is not a k -space.

Let the ground set of the spaces X and Y be

$(\omega\times\omega)\cup\{p\}$. The points of $\omega\times\omega$ in both spaces are isolated. For $f\in{}^{\omega}\omega$ and $n\in\omega$ put

$$U_f = \{(k,\ell)\in\omega\times\omega:\ \ell>f(k)\}\cup\{p\}$$

$$V^n = \{(k,\ell)\in\omega\times\omega:\ k>n\}\cup\{p\}\ .$$

A nbd -base of p in X is given by the system $\{U_f:f\in{}^{\omega}\omega\}$ while in Y by the system $\{V^n:n\in\omega\}$.

It is easily seen that X is a Frèchet-space and Y is first countable; we assert that $X\times Y$ is not a k -space.

Note first that if C is compact in X then $C\cap(\omega\times\omega)$ can be covered by a finite number of columns because a set $\{(k_n, \ell_n):n<\omega\}$, $k_n<k_{n+1}$ $(n<\omega)$ is an infinite closed discrete subset of X .

Similarly, if C is compact in Y then each column contains only finite points of C because for any $n<\omega$ $\{(n,k):k<\omega\}$ is a closed discrete subset of Y .

Put now

$$A = \Delta - \{(p,p)\} = \{(x,x):x\in\omega\times\omega\}\subset X\times Y\ .$$

A is not closed in $X\times Y$ because $(p,p)\in\overline{A}$ but for each compact subset K in $X\times Y$ the intersection $K\cap A$ is finite hence closed; $X\times Y$ is not a k -space. □

ad 8) By a theorem of Gruenhage [4] the product of a space with countable tightness and of a W -space has countable tightness. Moreover, Example 1 shows that the product with a G -space need not conserve the countable tightness.

The compact case

Let us now assume that one of the factor spaces is compact. We shall see that in this case the situation is totally different.

By a theorem of V. I. Malyhin [6] the product of two spaces with countable tightness one of which is compact, has countable tightness. This shows also that the product of a compact G-space and of a G-space is again a G-space. It is well-known (and easy to see) that the product of a k-space and of a compact space is a k-space. Finally, see [1] for a proof that the product of a compact sequential and of a sequential space is sequential.

Hence the only cases left open are the Fréchet spaces and the w-spaces. As for the Fréchet spaces, see P. Simon's example [8] for two compact Fréchet spaces whose product is not Fréchet. The other questions are solved by Example 3 and Theorems 1 and 2 below.

Example 3 A Fréchet space and a compact first countable space whose product is not Fréchet.

Let X be the Fréchet space X of Example 2 and let Y be $\omega+1$ (i.e. a convergent sequence with its limit). We assert that $Z=X\times Y$ is not Fréchet.

Put $A=\{((k,\ell),k):(k,\ell)\in\omega\times\omega\}$. Evidently $(p,\omega)\in\bar{A}-A$; we have to prove that no sequence $\{((k_n,\ell_n),k_n):n<\omega\}$ converges to (p,ω) in Z .

Indeed, if $((k_n,\ell_n),k_n)\to(p,\omega)$ then, by the topology of Y , $k_n\to\omega$ in Y but then the sequence $\{(k_n,\ell_n):n<\omega\}$ is closed in X hence does not converge to p . □

<u>Theorem</u> 1 The product of a compact Fréchet space and a w-space is Fréchet.

<u>Proof</u> Let X be a compact Fréchet space, Y a w-space, $Z=X\times Y$, $A\subset Z$, $(p,q)\in\bar{A}-A$. Put

$$T=\{x\in X: \text{there is a sequence } (x_n, y_n)\in A,\ (x_n, y_n)\to(x,q)\}.$$

We assert that $p\in\bar{T}$. Indeed, if U is any closed nbd of p in X, put $B=A\cap\pi_X^{-1}(U)$. As $(p,q)\in\bar{B}$ and Y is Fréchet, there is a sequence $(x_n, y_n)\in B$ with $y_n\to q$ in Y. Using now that X is compact and sequential hence also sequentially compact, we get a convergent subsequence $\{x_{n_k}:k<\omega\}$. If $\lim_k x_{n_k}=x$ then $x\in T\cap U$

Choose now a sequence $t_n\in T$ with $p=\lim t_n$. By the definition of T, for each $n\in\omega$ there is a sequence $\{(x_k^n, y_k^n):k<\omega\}\subset A$ with $\lim_k (x_k^n, y_k^n)=(t_n, q)$.

Let now $\omega=\cup\{T_i : i<\omega\}$ be a partition of ω into infinitely many infinite sets. For $(n,i)\in\omega\times\omega$ put

$$A_{n,i}=\{y_k^n : k\in T_i\}.$$

For any pair $(n,i)\in\omega\times\omega$ the sequence $A_{n,i}$ converges to the point q in Y. As Y is a w-space there is a sequence $\{k(n,i):(n,i)\in\omega\times\omega\}$ such that $k(n,i)\in T_i$ and $\{y^n_{k(n,i)}:(n,i)\in\omega\times\omega\}$ converges to q in Y.

It is now easily seen that the point p is in the closure of the set $\{x^n_{k(n,i)}:(n,i)\in\omega\times\omega\}$ in X. If S is an infinite subset of $\omega\times\omega$ such that the sequence $\{x^n_{k(n,i)}:(n,i)\in S\}$ converges to p in X then the subsequence $\{(x^n_{k(n,i)},y^n_{k(n,i)}):(n,i)\in S\}$ of A in $X\times Y$ converges to (p,q) . □

Theorem 2 The product of a compact w -space and of a w -space is again a w -space.

Proof Let X be a compact w -space, Y be a w -space, $Z=X\times Y$. By the last theorem Z is Fréchet, hence it is enough to prove that if $A_n\subset Z$ and A_n converges to the point (p,q) for each $n\in\omega$ then there is a sequence $z_n\in A_n$ with $z_n\to(p,q)$. Let $A_n=\{(x^n_k , y^n_k):k\in\omega\}$ and choose a partition $\{T_i : i\in\omega\}$ of ω into infinitely many pairwise disjoint infinite sets.

For $(n,i)\in\omega\times\omega$ put $B_{n,i}=\{y^n_k : k\in T_i\}$; evidently $B_{n,i}$ converges to q in Y for each $(n,i)\in\omega\times\omega$. Using now that Y is a w -space, select a function $k(n,i)\in T_i$ such that $\{y^n_{k(n,i)}:(n,i)\in\omega\times\omega\}$ converges to q in Y . Put $C_n=$ $=\{x^n_{k(n,i)}:i\in\omega\}$. Then for each $n\in\omega$ C_n converges to p in X . As X is a w -space, there is a sequence $\{i(n):n\in\omega\}$ such that $\{x^n_{k(n,i(n))}:n\in\omega\}$ converges to p in X . Note now that $z_n=(x^n_{k(n,i(n))},y^n_{k(n,i(n))})\in A_n$ and $z_n\to(p,q)$. □

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