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Ultrafilters without immediate predecessors in Rudin-Frolík order

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ULTRAFILTERS WITHOUT IMMEDIATE PREDECESSORS
IN RUDIN-FROLIK ORDER
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Abstract: We describe a construction of an ultrafilter on
the set of natural numbers not belonging into the closure of any
countable discrete set of minimal ultrafilters in Rudin-Frolík
order of \( \beta\mathbb{N} - \mathbb{N} \). We use the technique of independent linked family
developed by K. Kunen.

Key words: Ultrafilter, Rudin-Frolík order, independent
linked family, stratified set.

Classification: 04A20

§ 0. Introduction. Petr Simon has raised the following
question known as Simon's problem [1]: Does there exist a non-
minimal ultrafilter in Rudin-Frolík order of \( \beta\mathbb{N} - \mathbb{N} \) (shortly written
RF) without an immediate predecessor?

Let us call such an ultrafilter Simon point.

Two simple lemmas translate the property "being a Simon
point" into the topological terminology.

Lemma 0.1: An ultrafilter \( p \in \beta\mathbb{N} - \mathbb{N} \) is nonminimal in RF iff
there exists a countable discrete set \( X \subseteq \beta\mathbb{N} - \mathbb{N} \) of ultrafilters
such that \( p \in \overline{X - X} \).

Lemma 0.2: An ultrafilter \( p \in \beta\mathbb{N} - \mathbb{N} \) has an immediate prede­
cessor in RF iff there exists a countable discrete set \( X \) of
minimal ultrafilters in RF such that \( p \in \overline{X - X} \).

Therefore, Simon point \( p \) is an ultrafilter in \( \beta\mathbb{N} - \mathbb{N} \) for
which there exists a countable discrete set \( X \) such that \( p \in \overline{X - X} \).
and if \( Y \) is a countable discrete set of minimal ultrafilters in \( RF \) then \( p \notin \overline{Y} \).

The main result we want to present is the following

**THEOREM.** There exists a Simon point in \( \beta \mathbb{N} - \mathbb{N} \).

One can easily see that a Simon point \( p \) has to be in the closure of a countable discrete set of Simon points \( X_1 \). Since each point of \( X_1 \) is a Simon point, there exists a countable discrete set \( X_2 \) of Simon points such that \( X_1 \subseteq \overline{X_2} - X_2 \), and so on. Therefore, we shall construct countably many countable discrete sets \( X_n, n \in \mathbb{N} \) of Simon points such that \( X_n \subseteq \overline{X_{n+1}} - X_{n+1} \).

The original proof of Theorem needed the assumption that every set of functions from \( \omega^\omega \) of cardinality smaller than \( \mathbb{X} \) is bounded modulo fin. We are grateful to Petr Simon who has suggested us to use Kunen technique of independent linked family \([3]\) to avoid this assumption.

We would like also to thank Lev Bukovský for his manifold help and encouragement.

§ 1. **Preliminaries.** We shall use the standard notation and terminology to be found e.g. in \([4],[1]\). If \( \mathcal{G} \) is a filter then \( \mathcal{G} \) is the dual ideal. If \( G \) is a centered system of sets then \( (G) \) denotes a filter generated by this system. \( F \) refers to the Fréchet filter.

**Definition 1.1:** due to K.Kunen \([1]\). Let \( \mathcal{F} \) be a filter on \( \mathbb{N} \) and \( \mathcal{F} \supseteq \mathcal{F} \). \( A_n \subseteq \mathbb{N} \).

a) Let \( 1 \leq n < \omega \). An indexed family \( \{ A_\gamma ; \gamma \in [J]^n \} \) is precisely \( n \)-linked with respect to (w.r.t.) \( \mathcal{G} \) iff for all \( \sigma \in [J]^n \), \( \cap_{\gamma \in \sigma} A_\gamma \neq \emptyset \) and for all \( \sigma \in [J]^n \), \( \cap_{\gamma \in \sigma} A_\gamma \) is finite.

b) An indexed family \( \{ A_\gamma ; \gamma \in [J], n \in \omega \} \) is a linked
system w.r.t. \( \mathcal{F} \) iff for each \( \nu, \{ A_{\gamma} \colon \gamma \in \mathcal{J} \} \) is precisely \( \nu \)-linked w.r.t. \( \mathcal{F} \), and for each \( \nu \) and \( \gamma \), \( A_{\gamma} \subseteq \{ A_{\gamma} \colon \gamma \in \mathcal{J} \} \).

o) An indexed family \( \{ A_{\gamma}^{I} \colon \gamma \in \mathcal{J}, \xi \in I, \nu \in \omega \} \) is a \( \mathcal{J} \) by \( I \) independent linked family (ILF) w.r.t. \( \mathcal{F} \) iff for each \( \xi \in I \), \( \{ A_{\gamma}^{I} \colon \gamma \in \mathcal{J}, \nu \in \omega \} \) is a linked system w.r.t. \( \mathcal{F} \), and

\[ \bigcap_{\nu \in \omega} \bigcap_{\xi \in I} A_{\gamma}^{I} \notin \mathcal{F}^{I} \text{ whenever } \tau \in [I]^{<\omega}, \text{ and for each } \xi \in \tau, \nu \in \omega \text{ and } \sigma_{\xi} \in [J]^{<\omega} \].

Remark 1.2: If \( \{ A_{\gamma}^{I} \colon \xi \in I, \gamma \in \mathcal{J}, \nu \in \omega \} \) is independent linked family w.r.t. Prôchet filter. \( \mathcal{F} \supseteq \mathcal{F}, C \in \mathcal{F}, \tau \in [I]^{<\omega}, \sigma_{\xi} \in [J]^{<\omega} \) and \( B \supseteq \bigcap_{\nu \in \omega} \bigcap_{\xi \in I} A_{\gamma}^{I} \cap C \), then \( \{ A_{\gamma}^{I} \colon \xi \in \tau, \gamma \in \mathcal{J}, \nu \in \omega \} \) is independent linked family w.r.t. \( \mathcal{F} \cup \{ B \} \).

K.Kunen [3] has also proved the following

Proposition 1.3: There exists a \( 2^{\omega} \) by \( 2^{\omega} \) independent linked family w.r.t. Prôchet filter.

Definition 1.4: A countable set \( \{ \mathcal{F}_{\nu} \colon \nu \in \omega \} \) of filters on \( \omega \) is discrete iff there exists a partition of \( \omega \) (into disjoint sets) \( \{ A_{\nu} \colon \nu \in \omega \} \) such that \( A_{\nu} \in \mathcal{F}_{\nu} \) for each \( \nu \in \omega \).

Definition 1.5: A filter \( \mathcal{F} \) is in a closure of a discrete set of filters \( \{ \mathcal{F}_{\nu} \colon \nu \in \omega \} \) iff for each \( A \in \mathcal{F} \) the set \( \{ \nu \in \omega : A \in \mathcal{F}_{\nu} \} \) is infinite.

Definition 1.6: A set of filters \( \{ \mathcal{F}_{\nu, \mu} \colon \nu, \mu, \omega \} \) is stratified iff

1. the set \( \{ \mathcal{F}_{\nu, \mu} \colon \nu, \mu \in \omega \} \) is discrete for each \( \nu \in \omega \),
2. the filter \( \mathcal{F}_{\nu, \mu} \) is in the closure of the set \( \{ \mathcal{F}_{\nu, \ell} \colon \ell \in \omega \} \) for each \( \nu, \mu \in \omega \).

Definition 1.7: Let \( \{ \mathcal{F}_{\nu, \mu} \colon \nu, \mu \in \omega \} \) be a stratified set of filters and \( C \) be its subset. We define
\[ C(0) = \mathcal{C} \]
\[ C(\xi) = \bigcup_{\xi \in \mathcal{C}} C(\xi), \text{ if } \xi \text{ is limit.} \]
\[ C(\xi + 1) = C(\xi) \cup \{ F_{m, n} ; \exists B \in F_{m, n} \text{ such that} \]
\[ \{ F_{m + 1, n, \xi} ; B \in F_{m + 1, n, \xi} \} \subseteq C(\xi) \}
\[ \text{and } \mathcal{C} = \bigcup_{\xi < \omega_1} C(\xi). \]

We shall need the following result proved by M.E.Rudin [4].

**Lemma 1.8:** If \( X, Y \) are countable discrete sets of ultrafilters and \( \mu \in X \cap Y \) then \( \mu \in X \cap Y \cup X \cap (Y - Y) \cup Y \cap (X - X) \).

§ 2. Construction of a stratified set. The proof of
Theorem will be done via a construction of a stratified set
of ultrafilters with properties described in the following
proposition.

**Proposition 2.1:** There exists a stratified set of
ultrafilters \( \{ q_{m, n} ; m, n \in \omega \} \) on \( \omega \) satisfying for each
partition \( \{ D_\xi ; \xi \in \omega \} \) of \( \omega \) the following property (P):
Let \( C = \{ q_{m, n} ; (\exists \xi \in \omega)(D_\xi \in q_{m, n}) \} \). If \( q_{m, n} \notin \mathcal{C} \) then there
exists a family \( \{ U_\xi ; \xi \in \omega \} \subseteq q_{m, n} \) such that for each
\( \xi \in \omega \) and for each \( \xi_1 < \xi_2 < \ldots < \xi_i \), \( U_{\xi_1} \cap U_{\xi_2} \cap \ldots \cap U_{\xi_i} \cap D_\xi \)
is finite.

For to prove the proposition we need some auxiliary
results.

**Lemma 2.2:** If \( \{ F_{m, n} ; m, n \in \omega \} \) is a stratified set
of filters, \( \mathcal{A} = \{ A_I \ ; \ I \in \omega, I \supseteq \gamma \} \subseteq \omega \)
is II \( F \) w.r.t. \( F_{m, n} \) for every \( m, n \in \omega \) and \( B \subseteq \omega \) then there exists
a stratified set of filters \( \{ F_{m, n} ; m, n \in \omega \} \) and
\( \mathcal{C} = \{ A^i_j ; \{ i \in I, \gamma < 2^\omega, \delta < \omega \} \} \) an ILF w.r.t. \( \mathcal{F}_{\nu, \mu} \) for each \( \nu, \mu \in \omega \) such that \( \mathcal{F}_{\nu, \mu} \supseteq \mathcal{F}_{\nu_0, \mu} \), \( B \) or \( \omega - B \) belongs into \( \mathcal{F}_{\nu, \mu} \), \( \nu \leq \mu \) and \( \nu - \mu \) is countable.

**Proof.** Let us consider the set

\[
\mathcal{C} = \{ \mathbb{F}_{\alpha, j} ; \alpha \text{ is not ILF w.r.t. } (\mathbb{F}_{\alpha, j} \cup \{ B \}) \}.
\]

If \( \mathbb{F}_{\alpha, j} \) belongs to the set \( \mathcal{C} \) then there exist sets \( \mathbb{A}_{\alpha, j} \in [\Gamma]^\omega \) and \( E \in \mathbb{F}_{\alpha, j} \) such that \( B \cap E \cap \bigcap_{i \in \mathbb{A}_{\alpha, j}} A^i_{\gamma_i, j} = \emptyset \), i.e.

\[
\omega - B \supseteq E \cap \bigcap_{i \in \mathbb{A}_{\alpha, j}} A^i_{\gamma_i, j}.
\]

Evidently \( \{ A^i_{\gamma_i, j} ; \{ i \in I - \mathbb{A}_{\alpha, j}, \gamma < 2^\omega, \delta < \omega \} \} \) is ILF w.r.t. \( (\mathbb{F}_{\alpha, j} \cup \{ \omega - B \}) \).

We denote \( \mathcal{C} = \mathcal{F} = \mathcal{F}_{\alpha, j} \cup \{ \omega - B \} \) for \( \mathbb{F}_{\alpha, j} \in \mathcal{C} \). If \( \mathbb{F}_{\alpha, j} \notin \mathcal{C} \) then \( \mathbb{F} \) is ILF w.r.t. \( (\mathbb{F}_{\alpha, j} \cup \{ B \}) \).

It remains to show that \( \mathcal{C} \) is ILF w.r.t. \( (\mathbb{F}_{\alpha, j} \cup \{ \omega - B \}) \) if \( \mathbb{F}_{\alpha, j} \in \mathcal{C} - \mathcal{C} \). Suppose the opposite in order to get a contradiction. Let \( \beta \) be the least ordinal such that \( \mathbb{F}_{\alpha, \beta} \in \mathcal{C}(\beta) \) and \( \mathcal{C} \) is not ILF w.r.t. \( (\mathbb{F}_{\alpha, \beta} \cup \{ \omega - B \}) \).

Hence there exist sets \( E \in \mathbb{F}_{\alpha, \beta} \) and \( \tau \in [\Gamma]^\omega \) satisfying \( E \cap (\omega - B) \cap \bigcap_{i \in \tau} A^i_{\gamma_i, \beta} = \emptyset \). Take \( \mathbb{F}_{\alpha, \tau, \beta} \) containing \( E \) and \( \mathbb{F}_{\alpha, \tau, \beta} \in \mathcal{C}(\beta - 1) \). There exists such a filter. Then \( \mathcal{C} \) is not ILF w.r.t. \( (\mathbb{F}_{\alpha, \tau, \beta} \cup \{ \omega - B \}) \). This is a contradiction with the minimality of \( \beta \).

According to the foregoing discussion we denote

\[
\mathcal{C}_{\mu, \lambda} = \begin{cases} 
\mathcal{F}_{\mu, \lambda} \cup \{ \lambda \} & \text{for } \mathcal{F}_{\mu, \lambda} \notin \mathcal{C} \\
\mathcal{F}_{\mu, \lambda} \cup \{ \omega - \lambda \} & \text{otherwise.}
\end{cases}
\]
Lemma 2.2: If \( \{ \mathcal{F}_{n,m} ; n, m \in \omega \} \) is a stratified set of filters, \( \mathcal{A} = \{ A_{\gamma}^I \cap D_{\omega} ; \gamma < \omega^\omega, \langle \omega, \beta \in \omega \} \) is \( I \)-\( L \)-\( F \) w.r.t. \( \mathcal{F}_{n,m} \) for each \( n, m \in \omega \) and \( \mathcal{D} = \{ D_\gamma ; \gamma \in \omega \} \) is a partition of \( \omega \) such that \( D_\gamma \) or \( \omega - D_\gamma \) belongs into \( \mathcal{F}_{n,m} \) then there exists a stratified set of filters \( \{ \mathcal{F}_{n,m} ; n, m \in \omega \} \) and \( \mathcal{A} = \{ A_{\gamma}^I \cap D_{\omega} ; \gamma < \omega^\omega, \langle \omega, \beta \in \omega \} \) an \( I \)-\( L \)-\( F \) w.r.t. \( \mathcal{F}_{n,m} \) for each \( n, m \in \omega \) such that \( \mathcal{F}_{n,m} \supseteq \mathcal{F}_{n,m} \) and \( \mathcal{F}_{n,m} \) possesses the property \( (P) \) for the partition \( \mathcal{D} \), \( \hat{I} \subseteq I \) and \( I - \hat{I} \) is finite.

Proof: Let us consider the set
\( \mathcal{C} = \{ \mathcal{F}_{j,\xi} ; (\exists \gamma \in \omega)(D_\gamma \in \mathcal{F}_{j,\xi}) \} \).
If \( \mathcal{F}_{j,\zeta} \in \mathcal{C} \) we put \( \mathcal{F}_{j,\zeta} = \mathcal{F}_{j,\zeta} \).

Let \( \mathcal{F}_{j,\zeta} \notin \mathcal{C} \). Take \( \xi \in I \) and define (similarly as K.Kunen does)
\[ U_\gamma = \bigcup_{\xi \in \omega} (A_{\xi}^I \cap D_{\omega} \cup \gamma < \omega^\omega) \]
and \( \mathcal{F}_{j,\zeta} = \{ U_\xi ; \gamma < \omega^\omega \} \).
\[ U_\gamma \supseteq A_{\xi}^I \cap D_{\omega} \cup \gamma < \omega^\omega, \]
\( \mathcal{C} \) is \( I \)-\( L \)-\( F \) w.r.t. \( \mathcal{F}_{j,\zeta} \).

To verify the property \( (P) \), let \( \beta_1 < \beta_2 < \ldots < \beta_\delta < 2^\omega \).

The set \( U_{\beta_1} \cap U_{\beta_2} \cap \ldots \cap U_{\beta_\delta} \cap D_\zeta \) is a subset of
\[ A_{\beta_1}^I \cap A_{\beta_2}^I \cap \ldots \cap A_{\beta_\delta}^I \cap D_\zeta \]
which is in fact finite.

The set \( \{ \mathcal{F}_{n,m} ; n, m \in \omega \} \) is stratified by the definition of \( \mathcal{C} \).

q.e.d.

Proof of Proposition 2.1. We construct ultrafilters
\( \mathcal{Q}_{n,m} \) by the transfinite induction in \( 2^\omega \) stages.
At each stage \( \zeta < 2^\omega \) we will construct filters \( \mathcal{F}_{n,m} \).
and \( \mathcal{F}_{m,m} = \bigcup_{n \leq \omega} \mathcal{F}_{m,m}^{\leq} \). At the even stages we ensure that \( \mathcal{F}_{m,m} \) becomes ultrafilters and at the odd stages we ensure that \( \mathcal{F}_{m,m} \) will not belong into the closure of any countable discrete set of minimal ultrafilters. Simultaneously, at each stage we ensure that \( \mathcal{F}_{m,m} \) will belong into the closure of the set \( \{ \mathcal{F}_{m+1,\ell} ; \ell \in \omega \} \).

Let \( \{ B_{\ell} ; \ell \leq \omega, \ell \text{ even} \} \) enumerate all subsets of \( \omega \) and \( \{ D_{\ell} ; \ell \leq \omega, \ell \text{ odd} \} \) enumerate all partitions of \( \omega \). Let \( \mathcal{D}_\omega = \{ D_{\ell} ; \ell \in \omega \} \), in such a way that each partition occurs \( \ell^\omega \) many times in this enumeration.

Let \( \{ A_{\ell}^{\mathcal{D}_\omega} ; \ell < \omega, \mathcal{D} < \mathcal{D}_\omega, \mathcal{D} < \omega \} \) be independent linked family w.r.t. Fréchet filter \( \mathcal{F} \).

For each \( \mathcal{D} \), the system \( \{ A_{\ell}^{\mathcal{D} \mathcal{F}} ; \ell < \omega \} \) is almost disjoint. Put \( B_{m,m} = A_{m,m}^f \cup \bigcup_{\ell < \omega} A_{m,m}^{\mathcal{D} \ell} \). Let \( \{ C_{m} ; m \in \omega \} \) be a fixed partition of \( \omega \) on infinite sets. Suppose \( B_{m,m} \) is defined for each \( m \in \omega \). Put \( B_{m,m} = B_{m,m} \cap (A_{m,m}^f \cup \bigcup_{\ell < \omega} A_{m,m}^{\mathcal{D} \ell}) \) iff \( m \in C_\ell \). For each \( m \in \omega \), the system \( \{ B_{m,m} ; m \in \omega \} \) is pairwise disjoint.

Let \( \mathcal{G}_{m,m}^0 \) be a filter generated by \( \mathcal{F} \cup \{ B_{m,m} \} \cup \{ \omega - B_{m+1,\ell} ; \ell \in \omega \} \) for each \( m, m \in \omega \) and \( I_0 = 2^{\omega} - \omega \).

The set \( \{ A_{\ell}^{\mathcal{D} \mathcal{F}} ; \ell \in I_0, \ell < \omega, \mathcal{D} < \omega \} \) is ILF w.r.t. \( \mathcal{G}_{m,m}^0 \) for all \( m, m \in \omega \) according to Remark 1.2. (For each \( \mathcal{D} \in \mathcal{G}_{m,m}^0 \) there exist \( \mathcal{G} \in \mathcal{F} \) and \( A_{\ell}^{\mathcal{D} \mathcal{F}} ; \ell < m+1 \) satisfying \( \mathcal{D} \supseteq \mathcal{G} \cap \bigcup_{\ell=m+1} A_{\ell}^{\mathcal{D} \mathcal{F}} \)). The system \( \{ \mathcal{G}_{m,m}^0 ; m, m \in \omega \} \) is evidently stratified.

By the induction on \( \ell \leq 2^{\omega} \) we construct filters \( \mathcal{G}_{m,m}^\infty \) and an indexed set \( I_\infty \) with following properties:

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1) If $\lambda$ is even, we put $G_{\alpha, m, n}^{\omega+1} = \overline{G_{\alpha, m, n}^{\omega}}$ and $I_{\alpha+1} = \overline{I}_{\alpha}$ (using Lemma 2.2 where $B = B_{\alpha}$).

2) If $\lambda$ is odd, $S_{\alpha} = \{D_{\alpha, \beta}; \beta \in \omega\}$ is a partition of $\omega$ and assume that:

(A) for each $\mu \in \omega$ there exists $\beta \subset \lambda$, $\beta$ even such that $D_{\alpha, \beta} = D_{\beta}$, $\lambda$ being the first odd ordinal with this property. Hence for each $\mu \in \omega$ we have $D_{\alpha, \mu} \in G_{\alpha, m, n}$ or $\omega - D_{\alpha, \mu} \in G_{\alpha, m, n}$.

Then we define $G_{\alpha, m, n}^{\omega+1} = \overline{G_{\alpha, m, n}}$, $I_{\alpha+1} = \overline{I}_{\alpha}$ (using Lemma 2.3 where $D_{\alpha} = D$).

If the condition (A) does not hold true, we simply set $G_{\alpha, m, n}^{\omega+1} = G_{\alpha, m, n}$ and $I_{\alpha+1} = I_{\alpha}$.

3) If $\lambda$ is a limit ordinal we set $G_{\alpha, m, n}^{\omega+1} = \bigcup_{\beta \in \omega} G_{\alpha, m, n}^{\beta}$ and $I_{\alpha} = \bigcap_{\beta \in \omega} I_{\beta}$.

Finally we put $A_{m, n} = \bigcup_{\lambda \in \omega} G_{\alpha, m, n}$.

It remains to show that the set $\{A_{m, n}; m, n \in \omega\}$ satisfies the property required in Proposition 2.1.

Clearly, this set is stratified.

Assume that $B$ is a partition of $\omega$. Since each partition of $\omega$ occurs $\omega^{\omega}$ many times in the enumeration $\{D_{\alpha, \mu}; \mu \in \omega; \lambda \text{ odd}\}$ there exists a sufficiently large odd $\lambda$ such that $B = B_{\lambda}$ and the condition (A) is fulfilled. Now, we denote $C = \{q_{m, n} \mid (\exists \mu \in \omega)(D_{\alpha, \mu} \in q_{m, n})\}$ if $q_{m, n} \notin C'$ and $q_{m, n} \notin \overline{C_{\omega}}$ where $C_{\omega} = \{G_{\lambda, m, n}^{\omega}; (\exists \mu \in \omega)(D_{\alpha, \mu} \in G_{\lambda, m, n}^{\omega})\}$ then the family $\{u_{\alpha}; \gamma \subset \omega\}$ used in the construction of $G_{\alpha, m, n}^{\omega+1}$ according to the proof of Lemma 2.3 is the family desired by the proposition. Thus it is easy to show that

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for \( q_{m,n} \notin \mathcal{C} \) also \( s_{m,n} \notin \mathcal{C} \).

In order to get a contradiction we suppose that there exists \( q_{m,n} \notin \mathcal{C} \) and \( s_{m,n} \in \mathcal{C}(\beta) \) where \( \beta \) is the first ordinal with this property. Clearly, \( \beta \neq 0 \). By the definition of \( \mathcal{C}(\beta) \), there exists \( B \in s_{m,n} \subseteq q_{m,n} \) such that \( B = \{ q_{\alpha+1}, \alpha \in \omega \} \subseteq \mathcal{C}(\beta-1) \). By the minimality of \( \beta \), each \( q_{\alpha+1}, \alpha \in \omega \) is an element of \( \mathcal{C} \).

This is a contradiction with the assumption of \( q_{m,n} \notin \mathcal{C} \).

q.e.d.

§ 3. Proof of the THEOREM. Now, we are ready to prove the main result. Theorem follows immediately from Proposition 2.1 and Lemma 3.1.

Lemma 3.1: If \( \{ q_{m,n} ; m,n \in \omega \} \) is a stratified set of ultrafilters with the property \( (P) \) (of Proposition 2.1) then each \( q_{m,n} ; m,n \in \omega \) is a Simon point.

Proof: Since the set \( \{ q_{m,n} ; m,n \in \omega \} \) is stratified, each \( q_{m,n} \) is a nonminimal ultrafilter.

It remains to show that \( q_{m,n} \notin \overline{D} \) whenever \( D = \{ j \in \omega \} \) is a countable discrete set of minimal ultrafilters in \( \mathcal{F} \), \( m,n \in \omega \). Let \( \{ D_j ; j \in \omega \} \) be a partition of \( \omega \) such that \( D_j \in f_j \) for each \( j \in \omega \). Let \( C \) be as in Proposition 2.1. We show that \( C \cap \overline{D} = \emptyset \). Clearly, \( C(0) \cap \overline{D} = \emptyset \).

We proceed by induction. Suppose that \( C(\zeta) \cap \overline{D} = \emptyset \) and there exist \( \zeta, \eta \in \omega \) such that \( q_{\zeta,\eta} \in C(\zeta+1) \cap \overline{D} \). By Definition 1.7 there exists a set \( B \in q_{\zeta,\eta} \) with property \( q_{\zeta,\eta+1} \). This means that \( q_{\zeta,\eta} \in C(\zeta) \cap X_{\zeta+1} \).

Hence \( C(\zeta) \cap X_{\zeta+1} \cap \overline{D} \neq \emptyset \). But, this is impossible by Lemma 0.1 and Lemma 1.8.
Thus, if \( q_k, \ell \in i \) then \( q_k, \ell \notin D \).

Assume now \( q_k, \ell \notin \widehat{F} \) and \( \{ U_\zeta ; \zeta \in 2^\omega \} \leq q_k, \ell \) be such that for each \( \zeta \in \omega \) and for each \( \zeta_1 \prec \zeta_2 \prec \ldots \prec \zeta_\delta \), \( U_{\zeta_1} \cap U_{\zeta_2} \cap \ldots \cap U_{\zeta_\delta} \cap D \) is finite (the existence of \( U_\zeta \) follows from the property \((P)\)). Then for each \( \zeta \) there exist at most \( \zeta - 1 \) values of \( \zeta \) for which \( U_\zeta \in j_\zeta \). Thus there exists an ordinal \( \zeta \) such that \( U_\zeta \notin j_\zeta \) for each \( \zeta \in \omega \). This yields \( q_k, \ell \notin \widehat{D} \).

q.e.d.

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