Commentationes Mathematicae Universitatis Carolinae

Eva Butkovičová

Ultrafilters without immediate predecessors in Rudin-Frolík order

Commentationes Mathematicae Universitatis Carolinae, Vol. 23 (1982), No. 4, 757--766

Persistent URL: http://dml.cz/dmlcz/106194

COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE
23,4 (1982)

ULTRAFILTERS WITHOUT IMMEDIATE PREDECESSORS IN RUDIN-FROLÍK ORDER

M. BUTKOVIČOVÁ

__Abstract__: We describe a construction of an ultrafilter on the set of natural numbers not belonging into the closure of any countable discrete set of minimal ultrafilters in Rudin-Frolík order of $\beta N-N$. We use the technique of independent linked family developed by K.Kunen.

__Key words__: Ultrafilter, Rudin-Frolík order, independent linked family, stratified set.

Classification: 04 A 20

---

§ 0. __Introduction__. Petr Simon has raised the following question known as Simon's problem [1] : Does there exist a non-minimal ultrafilter in Rudin-Frolík order of $\beta N-N$ (shortly written RF) without an immediate predecessor ?

Let us call such an ultrafilter Simon point.

Two simple lemmas translate the property "being a Simon point" into the topological terminology.

__Lemma 0.1__: An ultrafilter $p\in\beta N-N$ is nonminimal in RF iff there exists a countable discrete set $X\subseteq\beta N-N$ of ultrafilters such that $p\in\overline{X}-X$.

__Lemma 0.2__: An ultrafilter $p\in\beta N-N$ has an immediate predecessor in RF iff there exists a countable discrete set X of minimal ultrafilters in RF such that $p\in\overline{X}-X$.

Therefore, Simon point p is an ultrafilter in $\beta N-N$ for which there exists a countable discrete set X such that $p\in\overline{X}-X$

and if Y is a countable discrete set of minimal ultrafilters in RF then $p \notin \overline{Y}$.

The main result we want to present is the following

THEOREM. There exists a Simon point in $\beta N - N$.

One can easily see that a Simon point p has to be in the closure of a countable discrete set of Simon points X_1. Since each point of X_1 is a Simon point, there exists a countable discrete set X_2 of Simon points such that $X_1 \subseteq \overline{X_2} - X_2$, and so on. Therefore, we shall construct countably many countable discrete sets X_n, $n \in \omega$ of Simon points such that $X_n \subseteq \overline{X_{n+1}} - X_{n+1}$.

The original proof of Theorem needed the assumption that every set of functions from ${}^{\omega}\omega$ of cardinality smaller than $2^{\aleph_0}$ is bounded modulo fin. We are grateful to Petr Simon who has suggested us to use Kunen technique of independent linked family [3] to avoid this assumption.

We would like also to thank Lev Bukovský for his manifold help and encouragement.

§ 1. Preliminaries. We shall use the standard notation and terminology to be found e.g. in [4],[1]. If $\mathcal{F}$ is a filter then $\mathcal{F}^*$ is the dual ideal. If G is a centered system of sets then (G) denotes a filter generated by this system. F refers to the Fréchet filter.

Definition 1.1: due to K.Kunen [3]. Let $\mathcal{F}$ be a filter on N and $\mathcal{F} \supseteq F$, $A_\eta \subseteq N$.

a) Let $1 \leq n < \omega$. An indexed family $\{A_\eta ; \eta \in J\}$ is precisely n-linked with respect to (w.r.t.) $\mathcal{F}$ iff for all $\sigma \in [J]^n$, $\bigcap_{\eta\in\sigma} A_\eta \notin \mathcal{F}^*$, but for all $\sigma \in [J]^{n+1}$, $\bigcap_{\eta\in\sigma} A_\eta$ is finite.

b) An indexed family $\{A_{\eta n} , \eta \in J, n \in \omega\}$ is a linked system w.r.t. $\mathcal{F}$ iff for each n, $\{A_{\eta n}\ ;\ \eta \in J\}$ is precisely n-linked w.r.t. $\mathcal{F}$, and for each n and η, $A_{\eta n} \subseteq A_{\eta n+1}$.

c) An indexed family $\{A^{\xi}_{\eta n}\ ;\ \eta \in J,\ \xi \in I,\ n \in \omega\}$ is a J by I independent linked family (ILF) w.r.t. $\mathcal{F}$ iff for each $\xi \in I$, $\{A^{\xi}_{\eta n}\ ;\ \eta \in J,\ n \in \omega\}$ is a linked system w.r.t. $\mathcal{F}$, and $\bigcap_{\xi \in \tau}(\bigcap_{\eta \in \sigma_\xi} A^{\xi}_{\eta n_\xi}) \notin \mathcal{F}^*$ whenever $\tau \in [I]^{<\omega}$, and for each $\xi \in \tau$, $1 \leq n_\xi < \omega$ and $\sigma_\xi \in [J]^{n_\xi}$.

Remark 1.2: If $\{A^{\xi}_{\eta n}\ ;\ \xi \in I,\ \eta \in J,\ n \in \omega\}$ is independent linked family w.r.t. $\mathcal{F} \supseteq F$, $C \in \mathcal{F}$, $\tau \in [I]^{<\omega}$, $\sigma_\xi \in [J]^{\leq n_\xi}$ and $B \supseteq \bigcap_{\xi \in \tau}(\bigcap_{\eta \in \sigma_\xi} A^{\xi}_{\eta n_\xi}) \cap C$, then $\{A^{\xi}_{\eta n}\ ;\ \xi \in I - \tau,\ \eta \in J,\ n \in \omega\}$ is independent linked family w.r.t. $(\mathcal{F} \cup \{B\})$.

K.Kunen [3] has also proved the following

Proposition 1.3: There exists a 2^ω by 2^ω independent linked family w.r.t. Fréchet filter.

Definition 1.4: A countable set $\{\mathcal{F}_n\ ;\ n \in \omega\}$ of filters on ω is discrete iff there exists a partition of ω (into disjoint sets) $\{A_n\ ;\ n \in \omega\}$ such that $A_n \in \mathcal{F}_n$ for each $n \in \omega$.

Definition 1.5: A filter $\mathcal{F}$ is in a closure of a discrete set of filters $\{\mathcal{F}_n\ ;\ n \in \omega\}$ iff for each $A \in \mathcal{F}$ the set $\{n \in \omega\ ;\ A \in \mathcal{F}_n\}$ is infinite.

Definition 1.6: A set of filters $\{\mathcal{F}_{n,m}\ ;\ n, m \in \omega\}$ is stratified iff

(1) the set $\{\mathcal{F}_{n,m}\ ;\ m \in \omega\}$ is discrete for each $n \in \omega$,

(2) the filter $\mathcal{F}_{n,m}$ is in the closure of the set $\{\mathcal{F}_{n+1,\ell}\ ;\ \ell \in \omega\}$ for each $n, m \in \omega$.

Definition 1.7: Let $\{\mathcal{F}_{n,m}\ ;\ n, m \in \omega\}$ be a stratified set of filters and C be its subset. We define

$C(0)=C$

$C(\alpha)=\bigcup_{\beta<\alpha} C(\beta)$, if α is limit.

$C(\alpha+1)=C(\alpha)\cup\{\mathcal{F}_{n,m}\ ;\ \exists B\in\mathcal{F}_{n,m}$ such that

$\{\mathcal{F}_{n+1,\ell}\ ;\ B\in\mathcal{F}_{n+1,\ell}\}\subseteq C(\alpha)\}$

and $\tilde{C}=\bigcup_{\alpha<\omega_1} C(\alpha)$.

We shall need the following result proved by M.E.Rudin [4].

Lemma 1.8: If X, Y are countable discrete sets of ultrafilters and $p\in\overline{X}\cap\overline{Y}$ then $p\in \overline{X\cap Y}\cup\overline{X\cap(\overline{Y}-Y)}\cup\overline{Y\cap(\overline{X}-X)}$.

§ 2. Construction of a stratified set. The proof of Theorem will be done via a construction of a stratified set of ultrafilters with properties described in the following proposition.

Proposition 2.1: There exists a stratified set of ultrafilters $\{q_{n,m}\ ;\ n,m\in\omega\}$ on ω satisfying for each partition $\{D_i\ ;\ i\in\omega\}$ of ω the following property (P): Let $C=\{q_{n,m}\ ;(\exists i\in\omega)(D_i\in q_{n,m})\}$. If $q_{k,\ell}\notin\tilde{C}$ then there exists a family $\{U_\alpha\ ;\ \alpha\in 2^\omega\}\subseteq q_{k,\ell}$ such that for each $i\in\omega$ and for each $\alpha_1<\alpha_2<\ldots<\alpha_i$, $U_{\alpha_1}\cap U_{\alpha_2}\cap\ldots\cap U_{\alpha_i}\cap D_i$ is finite.

For to prove the proposition we need some auxiliary results.

Lemma 2.2: If $\{\mathcal{F}_{n,m}\ ;\ n,m\in\omega\}$ is a stratified set of filters, $\mathcal{A}=\{A^{\xi}_{\eta k}\ ;\ \xi\in I, |I|>\omega, \eta<2^\omega, k\in\omega\}$ is ILF w.r.t. $\mathcal{F}_{n,m}$ for every $n, m\in\omega$ and $B\subseteq\omega$ then there exists a stratified set of filters $\{\overline{\mathcal{F}}_{n,m}\ ;\ n,m\in\omega\}$ and

$\overline{\mathcal{A}} = \{A^{\xi}_{\eta k} ;\ \xi \in \overline{I},\ \eta < 2^{\omega},\ k \in \omega\}$ an ILF w.r.t. $\overline{\mathcal{F}_{n,m}}$ for each $n, m \in \omega$ such that $\overline{\mathcal{F}_{n,m}} \supseteq \mathcal{F}_{n,m}$, B or $\omega - B$ belongs into $\overline{\mathcal{F}_{n,m}}$, $\overline{I} \subseteq I$ and $I - \overline{I}$ is countable.

Proof. Let us consider the set

$$C = \{ \mathcal{F}_{i,j} ;\ \mathcal{A} \text{ is not ILF w.r.t. } (\mathcal{F}_{i,j} \cup \{B\})\} .$$

If $\mathcal{F}_{i,j}$ belongs to the set C then there exist sets $\tau_{i,j} \in [I]^{<\omega}$ and $E \in \mathcal{F}_{i,j}$ such that $B \cap E \cap \bigcap_{\xi \in \tau_{i,j}} \bigcap_{\eta \in \sigma_\xi} A^{\xi}_{\eta k_\xi} = \emptyset$, i.e. $\omega - B \supseteq E \cap \bigcap_{\xi \in \tau_{i,j}} \bigcap_{\eta \in \sigma_\xi} A^{\xi}_{\eta k_\xi}$.
Evidently $\{A^{\xi}_{\eta k} ;\ \xi \in I - \tau_{i,j},\ \eta < 2^{\omega},\ k \in \omega\}$ is ILF w.r.t. $(\mathcal{F}_{i,j} \cup \{\omega - B\})$.

We denote $\overline{I} = I - \bigcup\{\tau_{i,j} ;\ \mathcal{F}_{i,j} \in C\}$. Therefore, $\overline{\mathcal{A}} = \{A^{\xi}_{\eta k} ;\ \xi \in \overline{I},\ \eta < 2^{\omega},\ k < \omega\}$ is ILF w.r.t. $(\mathcal{F}_{i,j} \cup \{\omega - B\})$ for $\mathcal{F}_{i,j} \in C$. If $\mathcal{F}_{k,\ell} \notin \tilde{C}$ then $\overline{\mathcal{A}}$ is ILF w.r.t. $(\mathcal{F}_{k,\ell} \cup \{B\})$.

It remains to show that $\overline{\mathcal{A}}$ is ILF w.r.t. $(\mathcal{F}_{k,\ell} \cup \{\omega - B\})$ if $\mathcal{F}_{k,\ell} \in \tilde{C} - C$. Suppose the opposite in order to get a contradiction. Let β be the least ordinal such that $\mathcal{F}_{k,\ell} \in C(\beta)$ and $\overline{\mathcal{A}}$ is not ILF w.r.t. $(\mathcal{F}_{k,\ell} \cup \{\omega - B\})$. Hence there exist sets $E \in \mathcal{F}_{k,\ell}$ and $\tau \in [\overline{I}]^{<\omega}$ satisfying $E \cap (\omega - B) \cap \bigcap_{\xi \in \tau} \bigcap_{\eta \in \sigma_\xi} A^{\xi}_{\eta k_\xi} = \emptyset$. Take $\mathcal{F}_{k+1,t}$ containing E and $\mathcal{F}_{k+1,t} \in C(\beta - 1)$. There exists such a filter. Then $\overline{\mathcal{A}}$ is not ILF w.r.t. $(\mathcal{F}_{k+1,t} \cup \{\omega - B\})$. This is a contradiction with the minimality of β .

According to the foregoing discussion we denote

$$\overline{\mathcal{F}_{n,m}} = \begin{cases} (\mathcal{F}_{n,m} \cup \{B\}) & \text{for } \mathcal{F}_{n,m} \notin \tilde{C} \\ (\mathcal{F}_{n,m} \cup \{\omega - B\}) & \text{otherwise.} \end{cases}$$

Lemma 2.3: If $\{\mathcal{F}_{n,m} ; n,m \in \omega\}$ is a stratified set of filters, $\mathcal{A} = \{A^{\xi}_{\eta k} ; \xi \in I, \eta < 2^{\omega}, k < \omega\}$ is ILF w.r.t. $\mathcal{F}_{n,m}$ for each $n, m \in \omega$ and $\mathcal{D} = \{D_i ; i \in \omega\}$ is a partition of ω such that D_i or $\omega - D_i$ belongs into $\mathcal{F}_{n,m}$ then there exists a stratified set of filters $\{\widehat{\mathcal{F}}_{n,m} ; n,m \in \omega\}$ and $\widehat{\mathcal{A}} = \{A^{\xi}_{\eta k} ; \xi \in \widehat{I}, \eta < 2^{\omega}, k < \omega\}$ an ILF w.r.t. $\widehat{\mathcal{F}}_{n,m}$ for each $n, m \in \omega$ such that $\widehat{\mathcal{F}}_{n,m} \supseteq \mathcal{F}_{n,m}$, $\widehat{\mathcal{F}}_{n,m}$ possesses the property (P) for the partition $\mathcal{D}$, $\widehat{I} \subseteq I$ and $I - \widehat{I}$ is finite.

Proof: Let us consider the set

$C = \{\mathcal{F}_{j,\ell} ; (\exists i \in \omega)(D_i \in \mathcal{F}_{j,\ell})\}$.

If $\mathcal{F}_{s,t} \in \widetilde{C}$ we put $\widehat{\mathcal{F}}_{s,t} = \mathcal{F}_{s,t}$.

Let $\mathcal{F}_{s,t} \notin \widetilde{C}$. Take $\xi \in I$ and define (similarly as K.Kunen does)

$\mathcal{U}_{\eta} = \bigcup_{k \in \omega} (A^{\xi}_{\eta k} \cap D_{k+1})$, $\widehat{I} = I - \{\beta\}$

and $\widehat{\mathcal{F}}_{s,t} = (\mathcal{F}_{s,t} \cup \{\mathcal{U}_{\eta} ; \eta < 2^{\omega}\})$.

$\mathcal{U}_{\eta} \supseteq A^{\xi}_{\eta k} \cap \bigcap_{i \le k} (\omega - D_i)$, therefore $\widehat{\mathcal{A}}$ is ILF w.r.t. $\widehat{\mathcal{F}}_{s,t}$.

To verify the property (P), let $\beta_1 < \beta_2 < \ldots < \beta_i < 2^{\omega}$. The set $\mathcal{U}_{\beta_1} \cap \mathcal{U}_{\beta_2} \cap \ldots \cap \mathcal{U}_{\beta_i} \cap D_i$ is a subset of $A^{\xi}_{\beta_1 i-1} \cap A^{\xi}_{\beta_2 i-1} \cap \ldots \cap A^{\xi}_{\beta_i i-1}$ which is in fact finite.

The set $\{\widehat{\mathcal{F}}_{n,m} ; n,m \in \omega\}$ is stratified by the definition of $\widetilde{C}$.

q.e.d.

Proof of Proposition 2.1. We construct ultrafilters $q_{n,m}$, $n,m \in \omega$ by the transfinite induction in 2^{ω} stages. At each stage $\alpha < 2^{\omega}$ we will construct filters $\mathcal{F}^{\alpha}_{n,m}$

and $q_{n,m} = \bigcup_{\alpha<2^\omega} \mathcal{F}^\alpha_{n,m}$. At the even stages we ensure that $q_{n,m}$'s become ultrafilters and at the odd stages we ensure that $q_{n,m}$'s will not belong into the closure of any countable discrete set of minimal ultrafilters. Simultaneously, at each stage we ensure that $q_{n,m}$ will belong into the closure of the set $\{q_{n+1,\ell} ; \ell \in \omega\}$.

Let $\{B_\alpha ; \alpha<2^\omega, \alpha \text{ even}\}$ emmumerate all subsets of ω and $\{\mathcal{D}_\alpha ; \alpha < 2^\omega, \alpha \text{ odd}\}$ emmumerate all partitions of ω, $\mathcal{D}_\alpha = \{D_{\alpha i} ; i \in \omega\}$, in such a way that each partition occurs $2^{\aleph_0}$ many times in this emmumeration.

Let $\{A^\xi_{\eta k} ; \xi < 2^\omega, \eta < 2^\omega, k < \omega\}$ be independent linked family w.r.t. Fréchet filter F .

For each ξ , the system $\{A^\xi_{\eta 1} ; \eta<2^\omega\}$ is almost disjoint. Put $B_{1,m} = A^1_{m1} - \bigcup_{j<m} A^1_{j1}$. Let $\{C_n ; n \in \omega\}$ be a fixed partition of ω on infinite sets. Suppose $B_{n,m}$ is defined for each $m<\omega$. Put $B_{n+1,m} = B_{n,\ell} \cap (A^{n+1}_{m1} - \bigcup_{j<m} A^{n+1}_{j1})$ iff $m \in C_\ell$. For each $n \in \omega$, the system $\{B_{n,m} ; m \in \omega\}$ is pairwise disjoint.

Let $\mathcal{F}^0_{n,m}$ be a filter generated by $F \cup \{B_{n,m}\} \cup \cup \{\omega - B_{n+1,\ell} ; \ell \in \omega\}$ for each $n, m \in \omega$ and $I_0 = 2^\omega - \omega$.

The set $\{A^\xi_{\eta k} ; \xi \in I_0 , \eta < 2^\omega, k<\omega\}$ is ILF w.r.t. $\mathcal{F}^0_{n,m}$ for all $n , m \in \omega$ according to Remark 1.2. (For each $B \in \mathcal{F}^0_{n,m}$ there exist $G \in F$ and $A^j_{\eta_j 1}$, $j \le n+1$ satisfying $B \supseteq G \cap \bigcap_{j \le n+1} A^j_{\eta_j 1}$). The system $\{\mathcal{F}^0_{n,m} ; n, m \in \omega\}$ is evidently stratified.

By the induction on $\alpha < 2^\omega$ we construct filters $\mathcal{F}^\alpha_{n,m}$ and an indexed set I_α with following properties:

1) If α is even, we put $\mathcal{F}^{\alpha+1}_{n,m} = \overline{\mathcal{F}^{\alpha}_{n,m}}$ and $I_{\alpha+1} = \overline{I}_{\alpha}$ (using Lemma 2.2 where $B = B_{\alpha}$).

2) If α is odd, $\mathcal{D}_{\alpha} = \{D_{\alpha i} ; i \in \omega\}$ is a partition of ω and assume that:

(A) for each $i \in \omega$ there exists $\beta < \alpha$, β even such that $D_{\alpha i} = B_{\beta}$, α being the first odd ordinal with this property. Hence for each $i \in \omega$ we have $D_{\alpha i} \in \mathcal{F}^{\alpha}_{n,m}$ or $\omega - D_{\alpha i} \in \mathcal{F}^{\alpha}_{n,m}$.

Then we define $\mathcal{F}^{\alpha+1}_{n,m} = \widehat{\mathcal{F}^{\alpha}_{n,m}}$, $I_{\alpha+1} = \widehat{I}_{\alpha}$ (using Lemma 2.3 where $\mathcal{D}_{\alpha} = \mathcal{D}$).

If the condition (A) does not hold true, we simply set $\mathcal{F}^{\alpha+1}_{n,m} = \mathcal{F}^{\alpha}_{n,m}$ and $I_{\alpha+1} = I_{\alpha}$.

3) If α is a limit ordinal we set $\mathcal{F}^{\alpha}_{n,m} = \bigcup\limits_{\beta<\alpha} \mathcal{F}^{\beta}_{n,m}$ and $I_{\alpha} = \bigcap\limits_{\beta<\alpha} I_{\beta}$.

Finally we put $q_{n,m} = \bigcup\limits_{\alpha<2^{\omega}} \mathcal{F}^{\alpha}_{n,m}$.

It remains to show that the set $\{q_{n,m} ; n, m \in \omega\}$ satisfies the property required in Proposition 2.1.

Clearly, this set is stratified.

Assume that $\mathcal{D}$ is a partition of ω . Since each partition of ω occurs $2^{\aleph_0}$ many times in the enumeration $\{\mathcal{D}_{\alpha} ; \alpha \in 2^{\omega}, \alpha \text{ odd}\}$ there exists a sufficiently large odd α such that $\mathcal{D} = \mathcal{D}_{\alpha}$ and the condition (A) is fulfilled. Now, we denote $C = \{q_{k,\ell} ; (\exists i \in \omega)(D_{\alpha i} \in q_{k,\ell})\}$. If $q_{n,m} \notin \widetilde{C}$ and $\mathcal{F}^{\alpha}_{n,m} \notin \widetilde{C}_{\alpha}$ where $C_{\alpha} = \{\mathcal{F}^{\alpha}_{k,\ell} ; (\exists i \in \omega)(D_{\alpha i} \in \mathcal{F}^{\alpha}_{k,\ell})\}$ then the family $\{\mathcal{U}_{\eta} ; \eta < 2^{\omega}\}$ used in the construction of $\mathcal{F}^{\alpha+1}_{n,m}$ according to the proof of Lemma 2.3 is the family desired by the proposition. Thus it remains to show that for $q_{n,m} \notin \tilde{C}$ also $\mathcal{F}^{\alpha}_{n,m} \notin \tilde{C}_{\alpha}$.

In order to get a contradiction we suppose that there exists $q_{n,m} \notin \tilde{C}$ and $\mathcal{F}^{\alpha}_{n,m} \in C_{\alpha}(\beta)$ where β is the first ordinal with this property. Clearly, $\beta \neq 0$. By the definition of $C_{\alpha}(\beta)$, there exists $B \in \mathcal{F}^{\alpha}_{n,m} \subseteq q_{n,m}$ such that $\mathcal{B} = \{\mathcal{F}^{\alpha}_{n+1,e} ; B \in \mathcal{F}^{\alpha}_{n+1,e}\} \subseteq C_{\alpha}(\beta - 1)$. By the minimality of β, each $q_{n+1,e} \supseteq \mathcal{F}^{\alpha}_{n+1,e} \in \mathcal{B}$ is an element of $\tilde{C}$. This is a contradiction with the assumption of $q_{n,m} \notin \tilde{C}$.

q.e.d.

§ 3. Proof of the THEOREM. Now, we are ready to prove the main result. Theorem follows immediatelly from Proposition 2.1 and Lemma 3.1.

Lemma 3.1: If $\{q_{n,m} ; n,m \in \omega\}$ is a stratified set of ultrafilters with the property (P) (of Proposition 2.1) then each $q_{n,m}$; $n,m \in \omega$ is a Simon point.

Proof: Since the set $\{q_{n,m} ; n,m \in \omega\}$ is stratified, each $q_{n,m}$ is a nonminimal ultrafilter.

It remains to show that $q_{n,m} \notin \overline{\mathbb{D}}$ whenever $\mathbb{D} = \{j_i ; i \in \omega\}$ is a countable discrete set of minimal ultrafilters in RF, $n,m \in \omega$. Let $\{D_i ; i \in \omega\}$ be a partition of ω such that $D_i \in j_i$ for each $i \in \omega$. Let C be as in Proposition 2.1. We show that $\tilde{C} \cap \overline{\mathbb{D}} = \emptyset$. Clearly, $C(0) \cap \overline{\mathbb{D}} = \emptyset$. We proceed by induction. Suppose that $C(\alpha) \cap \overline{\mathbb{D}} = \emptyset$ and there exist $i, j \in \omega$ such that $q_{i,j} \in C(\alpha+1) \cap \overline{\mathbb{D}}$. By Definition 1.7 there exists a set $B \in q_{i,j}$ with property $\{q_{i+1,e} ; B \in q_{i+1,e}\} \subseteq C(\alpha)$. This means that $q_{i,j} \in \overline{C(\alpha) \cap X_{i+1}}$. Hence $\overline{C(\alpha) \cap X_{i+1}} \cap \overline{\mathbb{D}} \neq \emptyset$. But, this is imposible by Lemma 0.1 and Lemma 1.8.

Thus, if $q_{k,l} \in \widetilde{C}$ then $q_{k,l} \notin \overline{D}$.

Assume now $q_{k,l} \notin \widetilde{C}$ and $\{\mathcal{U}_\alpha ; \alpha \in 2^\omega\} \subseteq q_{k,l}$ be such that for each $i \in \omega$ and for each $\alpha_1 < \alpha_2 < \ldots < \alpha_i$, $\mathcal{U}_{\alpha_1} \cap \mathcal{U}_{\alpha_2} \cap \ldots \cap \mathcal{U}_{\alpha_i} \cap D_i$ is finite (the existence of $\mathcal{U}_\alpha$ follows from the property (P)). Then for each i there exist at most $i-1$ values of α for which $\mathcal{U}_\alpha \in j_i$. Thus there exists an ordinal α such that $\mathcal{U}_\alpha \notin j_i$ for each $i \in \omega$. This yields $q_{k,l} \notin \overline{D}$.

q.e.d.

R e f e r e n c e s

[1] L. Bukovský, E. Butkovičová: Ultrafilter with $\aleph_0$ predecessors in Rudin-Frolík order, Comment. Math. Univ. Carolinae 22(1981), 429-447.

[2] Z. Frolík: Sums of ultrafilters, Bull. Amer. Math. Soc. 73(1967), 87-91.

[3] K. Kunen: Weak P-points in N^* , preprint.

[4] M. E. Rudin: Partial orders of the types in βN, Trans. Amer. Math. Soc. 155(1971), 353-362.

Matematický ústav SAV

Februárového víťazstva 9, 041 54 Košice

Czechoslovakia

(Oblatum 2.8. 1982)