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ANOTHER NOTE ON CLOSED N-CELLS IN R^N

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Abstract: A question of Markl as to whether the region between two closed N-cells in R^N deformation retracts to a boundary component is answered affirmatively.

Key words: Obstruction theory, Čech cohomology, crumpled cube.

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In [11] Markl raised the following question. Let U and V be homeomorphic images of Int B^N such that $\overline{U} \cong B^N \cong \overline{V}$; then if $U \subset V$, does $\overline{V} - U$ deformation retract to Fr V ? In discussing this question, it is natural to assume that $\overline{U} \subset V$, in order to maintain links with the annulus conjecture (see closing remarks). Throughout this paper we will use the following notation: $A = \mathrm{Fr}\, U$, $B = \mathrm{Fr}\, V$, $R = \overline{V} - U$, $S = \overline{V} - \overline{U}$. Under our assumptions R is a generalized manifold with boundary $A \cup B$, and $S = R - A$.

Markl's question asks if R deformation retracts to A, or again to B. In addition we will discuss whether S deformation retracts to B.

Lemma 1. The spaces A,B,R and S are all locally compact ANRs.

Proof. This trivial for A and B (which are homeomorphic to S^{N-1}). To prove R is an ANR, it suffices to show that it is locally contractible, since it is clearly finite-dimensional [7;V 7.1] . Local contractibility is obvious if $x \in S \subset R$, since $\overline{V} \cong B^N$ is an ANR. If $x \in A$, let Q be an open neighbourhood of x in R. Since A is collared in $\overline{U}$, there is an open neighbourhood Q_+ of x in $\overline{V}$ which deformation retracts (via p say) to Q. (Such a Q_+ is obtained by addding to Q an open collar of $Q \cap A$ in $\overline{U}$) As $\overline{V}$ is locally contractible, there is an open neighbourhood P of x in $\overline{V}$ which contracts in Q_+, and so also in $Q = p(Q_+)$. By restriction to $P \cap R$, we obtain a contraction of the open neighbourhood $P \cap R$ of x inside Q, as required.

Finally S is an open subset of R, and so is an ANR by [7;III 7.9].

If $N \leq 2$, by the Schönflies and Jordan Curve Theorems it follows that $R \cong S^{N-1} \times [0,1]$, and both questions have affirmative answers. Henceforth we suppose $N \geq 3$, and so A and B are simply connected.

<u>Lemma 2.</u> In the above notation suppose $N \geq 3$. Then:-

(i) R is simply connected.

(ii) S is connected, and $\pi_1(S) \cong 1$ if A is collared in R.

Proof. As $\overline{U} \subset V$, A separates R^N into two domains, each with frontier A, and one of which contains the complement of V. This easily yields connectedness of R and S.

Since A is collared in $\overline{U} \cong B^N$, there exists an open subset R_+ of $\overline{V}$ which deformation retracts (along a collar) to R, and such that $R_+ \cap U \cong S^{N-1} \times (0,1)$. Applying van Kampen's theorem [6;p40] to R_+ and U, and recalling that $R_+ \cup U = \overline{V}$

is contractible, we obtain $\pi_1(R) \cong 1$. The argument to show $\pi_1(S) \cong 1$ if A is collared in R is similar.

Remarks. (1) Trivially B is collared in R, and so S is homotopy equivalent to the interior of the generalized manifold R.

(2) Example 3.3 of Fox and Artin [5] shows that S can be simply connected without A being collared in R. To modify their example to suit present needs, delete the interior of a small tame 3-cell from Z.

Theorem. In the above notation assume $N \geqslant 3$. Then:-

(i) R always deformation retracts to A (and also to B).

(ii) S deformation retracts to B if and only if $\pi_1(S) \cong 1$.

(iii) There always exists a retraction of S to B.

Remark. We will utilise obstruction theory for maps of ANRs (as opposed to the more familiar case of cell complexes). No convenient account of the theory in full generality exists, but the reader may consult [10] and also [8;pp193-195]. However the proof of the first two parts can also be completed using [7;VII 8.1] since in the simply connected case there is no call for use of local coefficients.

Proof of the theorem. The obstructions to the existence of a deformation retraction of R to B lie in $\check{H}^k(R,B;\widetilde{\pi_k(R,B)})$, where the coefficients are twisted by the action of $\pi_1(R)$ and the cohomology is Čech cohomology. By Lemma 2, $\pi_1(R) \cong 1$ and so the coefficients are in fact untwisted. By [11] and duality, for any coefficients G we have $\check{H}^k(R,A;G) \cong \check{H}^k(R,B;G) \cong 0$, and so the obstructions to deformation retraction to A and to B both vanish identically.

In (ii) the forward implication is trivial. The converse follows as above, since the obstructions lie in $\check{H}^k(S,B;\widetilde{\pi_k(S,B)})$ $(k=0,1..)$, and these groups vanish by the proof below of (iii). As for (iii), the obstructions lie in $\check{H}^k(S,B;\widetilde{\pi_{k-1}(B)})$ where again the coefficients are twisted. Since $B \cong S^{N-1}$, the coefficients and so the obstructions vanish if $k<N$. If $k>N$, since $\dim S = N$ the obstructions vanish by [2;II.15]. Finally $\pi_{N-1}(B) \cong \mathbb{Z}$, so the only remaining obstruction lies in $\check{H}^N(S,B;\widetilde{\mathbb{Z}})$. We finish the proof by showing this group is zero.

By H_* denote singular homology, which by [2;V.12.6] equals Borel-Moore homology with compact supports, denoted ${}_{BM}H^c_*$. By $\check{H}^*_c$ denote Čech (=sheaf) cohomology with compact supports. By duality [2;V.9.3] $H_1(S) \cong {}_{BM}H^c_1(S) \cong \check{H}^{N-1}_c(R^N,R^N - S)$; this in turn is isomorphic to $\check{H}^{N-2}_\varphi(R^N - S)$ by the exact sequence of a pair, where φ is the support family $c \cap (R^N-S)$. As $R^N - S$ is the disjoint union of $\overline{U}$ and $R^N - \overline{V}$, we find $\check{H}^{N-2}_\varphi(R^N-S) \cong \check{H}^{N-2}_{c\cap\overline{U}}(\overline{U}) \oplus \check{H}^{N-2}_{c\cap(R^N-\overline{V})}(R^N-\overline{V})$. The first summand vanishes since $\overline{U}$ is compact and contractible. By duality again the second summand is isomorphic to ${}_{BM}H^c_2(R^N,\overline{V})$, and this vanishes by the exact sequence of a pair since R^N and $\overline{V}$ are contractible. Thus $H_1(S) \cong 0$, and it follows by [6;p220] that $\pi_1(S)$ is perfect. Since a perfect group can only operate trivially on an abelian group, the coefficients are untwisted and the obstruction group is $\check{H}^N(S,B)$.

We finish the proof by showing that this group vanishes. Now:-

$$
\begin{aligned}
\check{H}^N(S,B) &\cong \check{H}^N(R^N - \overline{U}, R^N - V), && \text{by excision [2;II.12.5]}\\
&\cong {}_{BM}H_0^{cld|(V-\overline{U})}(V - \overline{U}), && \text{by duality [2.V.9.3]}\\
&\cong {}_{BM}H^c_0(V,\overline{U}) && \text{by excision[2;V.5.9]}
\end{aligned}
$$

and this last group vanishes by the exact sequence of a pair. To justify the first step, note that $(R^N - \overline{U}) - (V - \overline{U}) = R^N - V$. To justify the third step, note that if K is a closed subset of $R^N - \overline{U}$ lying in $V - \overline{U}$, then $K = L \cap (V - \overline{U})$ where $L \subset V \subset \overline{V} \subset R^N$ is closed, and so compact by the Heine-Borel theorem; thus the support families are as displayed. This finishes the proof.

Corollary. S deformation retracts to B if (but not only if) A is collared in R.

Remarks. It may amuse the reader to attempt to prove (iii) using engulfing and the fact that B is a neighbourhood retract.

Examples abound in which $\pi_1(S) \not\cong 1$. Let X be a crumpled cube, i.e. the closure of a complementary domain of an (N-1)-sphere in S^N. By the Hosay-Liniger-Daverman theorem (see [3;supplement 7] for a convenient reference), if $N \neq 4$, X can be re-embedded in S^N so that $S^N - X$ has closure homeomorphic to B^N. If we take U to be $S^N - X$ and V to be the complement of a tame closed N-cell in Int X, then by a van Kampen argument $\pi_1(S) \cong \pi_1(X - \text{Fr } X)$, which in general is non-trivial. For example, X could be the exterior of the Alexander horned sphere, embedded in S^3 in the usual way.

The results remind us of the need in the statement of the annulus conjecture for local flatness of A and B. By results of Bing [1;theorem 2] and Ferry [4;theorem 5] if A and B are 1-LC in R, then provided $N \neq 4$, they are flat. By the annulus conjecture ([9] if $N \geqslant 5$, [1] if $N = 3$) it follows that $R \cong A \times [0,1]$ if $N \neq 4$. The above results show that such lack of flatness may often appear in the fundamental group of a suitable subset.

Finally what happens if as in [11] we assume $U \subset V$, instead of $\overline{U} \subset V$? Lemma 1 is unchanged, but S may no longer be connected. The proofs of Lemma 2(i) and the first part of the theorem are unchanged in essence. In the second part of the theorem, the correct necessary and sufficient condition is now that S is simply connected. In the final part the conclusion is unaltered, but the proof is complicated as one must work component by component to construct the retraction.

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