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# A NOTE ON NORMABILITY OF LOCALLY CONVEX SPACES

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Abstract: Let X be a real or complex bornological and barrelled Hausdorff locally convex vector space and L(X) the algebra of all continuous linear operators of X into itself. It is proved that under certain purely algebraic conditions on L(X) there exists an inner product on X, such that X equipped with this inner product is a Hilbert space, and the topology induced by this inner product coincides with the given topology on X.

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Throughout this paper we denote by $L(X)$ the algebra of all continuous linear operators of a locally convex space $X$ into itself, by $F(X)$ the algebra of all continuous linear operators with finite dimensional range, and by $X^*$ the space of all continuous linear functionals acting on $X$. We shall write $x \otimes f$ for a continuous linear operator defined by the relation $(x \otimes f)y = f(y)x$ where $f \in X^*$ is a fixed functional and $x \in X$ a fixed vector. By involution we mean a linear and in the complex case a conjugate-linear mapping $A \mapsto A^*$ defined on $L(X)$ or $F(X)$ such that $(AB)^* = B^*A^*$ and $A^{**} = A$ is fulfilled. The

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purpose of this paper is to prove the following result.

Theorem 1. Let X be a real or complex bornological and barrelled Hausdorff locally convex space. Suppose that there exists an involution $A \mapsto A^*$ on $F(X)$ such that on some minimal left ideal $\mathcal{L} \subset F(X)$ the implication $A^*A = 0 \Rightarrow A = 0$ is fulfilled. In this case there exists an inner product $(\cdot,\cdot)$ on X such that X equipped with the inner product is a Hilbert space, and the topology induced by the inner product coincides with the given topology on X. For each $A \in F(X)$ the relation $(Ax,y) = (x,A^*y)$ is fulfilled for all pairs $x,y \in X$.

Remarks. In case X is a Banach space, the result above reduces to a theorem proved in our earlier paper [7]. The result above can be considered as an improvement of a well known result due to S. Kakutani and G.W. Mackey (see [3],[4],[1] and [6]) which characterises Banach spaces isomorphic to Hilbert spaces among all Banach spaces. Some results in the sense of Kakutani-Mackey theorem can be found in [7] and [8]. It should be mentioned that some results concerning the normability of the locally convex space X in terms of L(X) have been proved by J.H. Williamson [9]. For the proof of Theorem 1 we need the following lemma. We shall omit the proof of this lemma, since it is possible to use the unchanged proof of Lemma (4.10.1) in [5].

Lemma 1. Let $\mathcal{A}$ be an arbitrary $*$-algebra and $\mathcal{L} \subset \mathcal{A}$ a minimal left ideal. If for each $a \in \mathcal{L}$, $a^*a = 0$ implies $a = 0$, then $\mathcal{L}$ is of the form $\mathcal{L} = \mathcal{A}p$, where p is a unique hermitian idempotent.

Proof of Theorem 1. According to Lemma 1 there exists a unique hermitian idempotent $P$ such that the minimal left ideal $\mathcal{L}$ can be expressed in the form $\mathcal{L} = F(X)P$. Using the fact that the left ideal $\mathcal{L}$ is minimal, it is easy to prove that the range of the idempotent $P$ is one-dimensional, which allows us to introduce an inner product into $\mathcal{L}$ as follows:

$$(A,B)P = B^*A, \quad A,B \in \mathcal{L} \,. \tag{1}$$

Obviously, $(.,.)$ is linear in the first variable for the fixed second. Using the fact that $P^* = P$, we obtain $(A,B) = \overline{(B,A)}$. Since $(A,A)P = A^* A$ the implication $(A,A) = 0 \Rightarrow A = 0$ is fulfilled, whence it follows that $(.,.)$ is positive or negative definite. But $(P,P) = 1$, which implies that $(.,.)$ is positive definite. Therefore $(.,.)$ is indeed an inner product on $\mathcal{L}$. The idempotent $P$ can be expressed in the form

$$P = e \otimes f \tag{2}$$

where $e$ is a fixed nonzero vector, and $f \in X^*$ a fixed continuous linear functional such that $f(e) = 1$. Obviously, the minimal left ideal $\mathcal{L}$ contains exactly those operators which can be written in the form $x \otimes f$, where $f$ is the functional from (2), and $x \in X$ an arbitrary vector. The isomorphism $x \mapsto x \otimes f$ allows us to introduce an inner product into $X$ as follows:

$$(x,y) = (x \otimes f, y \otimes f), \quad x,y \in X \,. \tag{3}$$

Let us prove that

$$(Ax,y) = (x,A^*y) \tag{4}$$

holds for each $A \in F(X)$ and all pairs $x,y \in X$. Using the relation $(Ax) \otimes f = A(x \otimes f)$, we obtain $(Ax,y)P = ((Ax) \otimes f, y \otimes f)P =$ $= (A(x \otimes f), y \otimes f)P = (y \otimes f)^* A(x \otimes f)$. On the other hand,

$(x,A^*y)P = (x\otimes f,(A^*y)\otimes f)P \cdot (x\otimes f, A^*(y\otimes f))P =$
$= (A^*(y\otimes f))^*(x\otimes f) = (y\otimes f)^* A(x\otimes f).$

Now we intend to prove that to each $y\in X$ there corresponds a continuous seminorm $p(.)$ such that

$$(5) \qquad |(x,y)| \leq p(x)$$

holds for all $x\in X$. Let $A,B\in F(X)$ be arbitrary. Then for each continuous seminorm $p(.)$ there exists some continuous seminorm $q(.)$ such that

$$(6) \qquad p(ABx)\leq q(Bx)$$

is fulfilled for all $x\in X$. Let P, e and f be from (2). There exists a continuous seminorm $p(.)$ such that $p(e)\neq 0$ (recall that X is by assumption Hausdorff). Using (3) and (6) we obtain $|(x,y)|\,p(e) = p((x,y)Pe) = p((y\otimes f)^*(x\otimes f)e)\leq q((x\otimes f)e) =$ $= q(x)$. Hence $|(x,y)|\leq p(e)^{-1}q(x)$ which proves the relation (5).

The relation (5) means that for each $y\in X$ the linear functional $f_y$, defined by $f_y(x) = (x,y)$, is continuous. Let us prove the converse. More precisely, we intend to show that each continuous linear functional $g\in X^*$ can be written in the form $g(x) =$ $= (x,y)$ for some fixed $y\in X$. Let therefore $g\in X^*$ be given, and let us choose a fixed vector $u\in X$ such that $(u,u) = 1$. Using the relation (4) we obtain $g(x) = ((u\otimes g)x,u) = (x,(u\otimes g)^* u)$.

Let us now prove that the set $\mathcal{M}\subset X$ is bounded if and only if it is bounded with respect to the inner product. For this purpose we shall first prove that for each bounded set $\mathcal{M}\subset X$ there exists a continuous seminorm $p(.)$ such that

$$(7) \qquad |(x,y)|\leq p(x)$$

holds for all $x\in X$ and all $y\in\mathcal{M}$. Let therefore a bounded set $\mathcal{M} \subset X$ be given, and let us consider the family of continuous linear functionals $\{f_y;\ f_y(x) = (x,y),\ y \in \mathcal{M}\}$. By the relation (5) there corresponds for each $x \in X$ a continuous seminorm $p_x(.)$ such that $|f_y(x)| = |(x,y)| = |(y,x)| \leq \leq p_x(y)$. Using the inequality $|f_y(x)| \leq p_x(y)$ and the fact that the set $\mathcal{M}$ is bounded, we obtain $\sup_{y \in \mathcal{M}} |f_y(x)| < \infty$. Hence, since X is by assumption barrelled, there exists a continuous seminorm $p(.)$ such that $|f_y(x)| \leq p(x)$ holds for all $y \in \mathcal{M}$ and all $x \in X$, which proves (7). From (7) it follows that each bounded set is bounded also with respect to the inner product. It remains to prove the converse. Let us therefore assume that the set $\mathcal{M} \subset X$ is such that $(x,x) \leq C$ is fulfilled for all $x \in \mathcal{M}$ and some constant C. Let $f \in X^*$ be an arbitrary continuous linear functional. Using the fact that f can be expressed in the form $f(x) = (x,y)$ for some fixed $y \in X$, and the Schwartz inequality, we obtain $|f(x)| = |(x,y)| \leq (x,x)(y,y)$, whence $|f(x)| \leq C(y,y)$ for all $x \in \mathcal{M}$. Hence, for each $f \in X^*$ we have $\sup_{x \in \mathcal{M}} |f(x)| < \infty$, which implies that the set $\mathcal{M}$ is bounded (see [2, Theorem 3, p. 209]).

Therefore there are two topologies on X, the original one, and the topology induced by the inner product. Since we have just proved that the bounded sets in both topologies are the same, it follows that the identity mapping is in both directions continuous, since both topologies are bornological (see [2, Proposition 1, p. 220]). Hence, the topology induced by the inner product coincides with the given topology on X.

It remains to show that X equipped with the inner product is not only a pre-Hilbert but even a Hilbert space. This follows from the fact that a pre-Hilbert space, in which the Riesz representation theorem holds, is a Hilbert space. The proof of the theorem is complete.

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