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DISTRIBUTIVE GROUPOIDS AND PRERADICALS II

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Abstract: One-sided ideals and the corresponding preradicals of distributive groupoids are studied.

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This note is an immediate continuation of [2]. The theory of preradicals developed in [2] is applied to some special cases. Two preradicals derived from left and right ideals are defined and their rôle in the structure theory of distributive groupoids is studied.

9. Ideals. Let A denote the class of distributive idempotent groupoids.

9.1. Lemma. Let $G \in A$.

(i) If I is an ideal of G and K a left (right) ideal of I then K is a left (right) ideal of G.

(ii) If I is a left (right) ideal of G and K an ideal of I then K is a left (right) ideal of G.

(iii) If I is an ideal of G and K an ideal of I then K is an ideal of G.

Proof. (i) We have $ab \in I$ and $ab = ab.ab = (ab.a)(ab.b)$ for all $a \in G$ and $b \in K$. Since I is an ideal of G and K a left ideal of I, $ab.a \in I$ and $ab.b \in K$. Consequently, $ab \in K$.

(ii) We have $ab \in I$ and $ab = ab.ab = (a.ab)(b.ab)$ for all $a \in G$ and $b \in K$. Since I is a left ideal of G and K an ideal of I, $a.ab \in I$ and $b.ab \in K$. Consequently, $ab \in K$.

(iii) Use (i) or (ii).

9.2. <u>Lemma</u>. Let I, K be left ideals of a groupoid $G \in A$. Then IK is a left ideal of G and $IK \subseteq K$.

9.3. <u>Lemma</u>. Let I be a left and K a right ideal of a groupoid $G \in A$. Then $KI = I \cap K$ is a left (right) ideal of $K(I)$. Moreover, KI is an ideal of G, provided $IK \subseteq KI$.

9.4. <u>Lemma</u>. Let I and K be ideals of a groupoid $G \in A$. Then $IK = I \cap K = KI$ is an ideal of G.

9.5. <u>Lemma</u>. Let I be an ideal of a groupoid $G \in A$ and let $a,b,c \in G$. Then $ab \in I$ iff $ba \in I$ and $a.bc \in I$ iff $ab.c \in I$.

Proof. We have $ba = ba.ba = (ba.b)(ba.a) = (b.ab)(ba.a)$ and $ab.c = ac.bc = (a.bc)(c.bc)$, $a.bc = ab.ac = (ab.a)(ab.c)$.

For $G \in A$, define a relation $w(G)$ by $(a,b) \in w(G)$ iff the elements a and b generate the same ideal of G.

9.6. <u>Lemma</u>. Let $G \in A$. Then:

(i) $(a,b) \in w(G)$ iff $a = f(b)$ and $b = g(a)$ for some $f,g \in Mul(G)$.

(ii) $w(G)$ is a congruence of G, $H = G/w(G)$ is a semilattice and $w(H) = id_H$.

(iii) Every block of $w(G)$ is ideal-free.

(iv) If I is an ideal of G then $w(I) = w(G) \cap (I \times I)$.

Proof. (i) This is clear.

(ii) Apply (i) and 9.5.

(iii) Let H be a block of $w(G)$ and $a,b \in H$. There are positive integers n, m, $S_1,\dots,S_n$, $T_1,\dots,T_m \in \{L,R\}$ and $a_1,\dots,a_n$, $b_1,\dots,b_m \in G$ such that $a = S_{1,a_1} \dots S_{n,a_n}(b)$ and $b = T_{1,b_1} \dots T_{m,b_m}(a)$. Then $a = aa = S_{1,aa_1} \dots S_{n,aa_n}(a)$ and $b = T_{1,bb_1} \dots T_{m,bb_m}(ba)$. From this, it is easy to see that $aa_i, bb_j \in H$, and so $(a,b) \in w(H)$. Thus $w(H) = H \times H$ and H is ideal-free.

9.7. Corollary. w is an idempotent radical.

9.8. Lemma. Let I be an ideal of a groupoid $G \in A$. Then G is isomorphic to a subgroupoid of the product of G/I and a set of copies of I.

Proof. For every $a \in I$, both L_a and R_a are homomorphisms of G into I and $r \cap \bigcap \ker(L_a) \cap \bigcap \ker(R_a) = id$, where $r = (I \times I) \cup id$.

9.9. Corollary. Let I be an ideal of a groupoid $G \in A$. Then the groupoids G and $I \times G/I$ generate the same groupoid variety.

9.10. Proposition. Let $G \in A$. Then $w(G)$ is just the least congruence of G such that the corresponding factorgroupoid is a semilattice.

Proof. Denote by r the congruence. By 9.6(ii), $r \subseteq w(G)$. However, if H is a block of $w(G)$ then $r|H = H \times H$ as it follows from 9.6(iii). Hence $w(G) \subseteq r$.

9.11. Corollary. $w = m_S$, S being the class of semilattices (see 6.2).

10. **Left and right ideals.** Let A denote the class of distributive idempotent groupoids. A groupoid G is said to be left (right) permutable if it satisfies the identity $x.yz = y.xz$ ($xy.z = xz.y$).

10.1. Lemma. Let $G \in A$ be left (right) permutable. Then G is medial.

Proof. For $a,b,c,d \in G$, $ab.cd = c(ab.d) = c(ad.bd) = c(b(ad.d)) = b(c(ad.d)) = b(ad.cd) = b(ac.d) = ac.bd$.

10.2. Lemma. Let $G \in A$ be both left and right permutable. Then G is a semilattice.

Proof. For $a,b,c \in G$, $a.bc = ab.ac = (a.ac)b = (ab)(ac.b) = (ac)(ab.b) = (a(ab.b))c = (ab.ab)c = ab.c$ and $ab = ab.a = a.ba = ba$.

10.3. Lemma. Let I be a left ideal of a groupoid $G \in A$ and let $a,b,c \in G$. Then $a.bc \in I$ iff $b.ac \in I$.

Proof. $a.bc = ab.ac = (a.ac)(b.ac)$.

For $G \in A$, define a relation $u(G)$ (resp. $v(G)$) by $(a,b) \in u(G)$ (resp. $v(G)$) iff the elements a and b generate the same left(resp. right) ideal of G.

10.4. Lemma. Let $G \in A$. Then:

(i) $(a,b) \in u(G)$ iff $a = f(b)$ and $b = g(a)$ for some $f,g \in Mull(G)$.

(ii) $u(G)$ is a congruence of G, $H = G/u(G)$ is left permutable and $u(H) = id_H$.

(iii) If K is a block of $u(G)$ then $K/u(K)$ is a semigroup of right zeros.

Proof. (i) This is clear.

(ii) $u(G)$ is a congruence and H is left permutable by (i) and 10.3. Denote by f the natural projection of G onto H. Let

$a,b \in G$ be such that $(f(a),f(b)) \in u(H)$. Then there are positive integers n, m and $a_1,\dots,a_n$, $b_1,\dots,b_m \in G$ with $(a_1(a_2(\dots(a_n a))),b) \in u(G)$ and $(b_1(b_2(\dots(b_m b))),a) \in u(G)$. From this, it is easy to see that $(a,b) \in u(G)$.

(iii) Let $a,b \in K$. There are positive integers n, m and $a_1,\dots,a_n$, $b_1,\dots,b_m \in G$ with $a = b_1(\dots(b_m b))$ and $b = a_1(\dots(a_n a))$. Then $a = b_1(\dots(b_m(a_1(\dots(a_n a)))))$ and $a = b_1(\dots(b_{i-1}((b_i b_{i+1})(\dots((b_i b_m)((b_i a_1)(\dots(b_i a_n . b_i a)))))))$ for every $1 \leq i \leq m$ and we see that $b_i a \in K$. On the other hand, $a = (b_1(\dots(b_m b)))a = (b_1 a)(\dots(b_m a . ba))$, and therefore $(a,ba) \in u(K)$.

10.5. Corollary. Both u and v are radicals.

10.6. Corollary. Both $\bar{u}$ and $\bar{v}$ are idempotent radicals.

10.7. Lemma. Let $G \in A$, $n \geq 2$ and $a_1,\dots,a_n \in G$. Then there are $b_1,\dots,b_{n-2} \in G$ such that $((a_1a_2)\dots)a_n = b_1(\dots(b_{n-2} . R_{a_n}^{n-1}(a_1)))$.

Proof. By induction on n. For $n = 2$, there is nothing to prove. For $n \geq 3$, $((a_1a_2)\dots)a_n = b_1(\dots(b_{n-3}c))$, $c = R_{a_n}^{n-2}(a_1a_2)$. But, $c = R_{a_n}^{n-2}(a_1).R_{a_n}^{n-2}(a_2) = R_{a_n}^{n-2}(a_1)(R_{a_n}^{n-3}(a_2).\ a_n) = (R_{a_n}^{n-2}(a_1).R_{a_n}^{n-3}(a_2)).R_{a_n}^{n-1}(a_1)$.

10.7. Lemma. Let $G \in A$, $n \geq 1$, $a,b,a_1,\dots,a_n \in G$, $a = ((ba_1)\dots)a_n$ and let H be the block of $u(G)$ containing a. Then there are $m \geq 1$ and $b_1,\dots,b_m \in H$ such that $a = ((bb_1)\dots)b_m$.

Proof. Let $1 \leq i \leq n$. We have $a = c(((a_i a_{i+1})\dots)a_n)$, $c = (((((ba_1)\dots)a_{i-1})a_{i+1})\dots)a_n$. From this, $a = aa = (ca)((((a_i a_{i+1})\dots)a_n)a)$. By 10.6, there are $c_1,\dots,c_{n-i} \in G$ such that $a = (ca)(c_1(\dots(c_{n-i} . R^{n-i+1}(a_i))))$. Obviously,

$R_a^{n-i+1}(a_i) \in H$. However, then $d_i = R_a^n(a_i) \in H$ and $a =$
$= ((R_a^n(b)d_1)\dots)d_n$.

10.8. Proposition. (i) $u.v. = v.u = u \cap v$.

(ii) $\widehat{a\ell} \subseteq \bar{u}$ and $\widehat{ar} \subseteq \bar{v}$ (see 7.2).

(iii) $m_M \subseteq u \cap v$ (see 7.3).

(iv) $u+v \subseteq w$, $u:v \subseteq w$ and $v:u \subseteq w$.

Proof. (i) By 10.7 and its dual, $u \cap v \subseteq v.u$, $u \cap v \subseteq u.v$ and the result follows from 4.1(i).

(ii) The inclusion $a\ell \subseteq u$ is clear directly from the definitions. Since u is a radical and $\widehat{a\ell}$ is idempotent, $\widehat{a\ell} \subseteq \bar{u}$.

(iii) This follows from 10.1, 10.4(ii) and its dual.

(iv) This is clear.

10.9. Corollary. $u^n.v^m = v^m.u^n = u^n \cap v^m$ for all positive integers n, m.

10.10. Corollary. $u.\bar{v} \subseteq \bar{v}.u$ and $v.\bar{u} \subseteq \bar{u}.v$.

10.11. Lemma. Let $G \subset A$ be left permutable and $(a,b) \in u(G)$. Then $ab = b$ and $ba = a$.

10.12. Proposition. Let $G \subset A$ be left permutable. Then:

(i) $u(G) \subseteq ar(G) \subseteq v(G) = w(G) = \bar{v}(G)$.

(ii) Every block of $u(G)$ is a semigroup of right zeros.

(iii) $\bar{u}(G) = u^2(G) = id_G$ and $a\ell(G) = id_G$.

Proof. (i) By 10.11 and 10.8(ii),(iv), $u(G) \subseteq ar(G) \subseteq v(G) \subseteq$ $\subseteq w(G)$. Denote by f the natural projection of G onto $H = G/v(G)$. By 10.2 and the dual of 10.4(ii), H is a semilattice, and hence $w(H) = id_H$. If $(a,b) \in w(G)$ then $(f(a),f(b)) \in w(H)$, $f(a) = f(b)$ and $(a,b) \in v(G)$. Thus $w(G) = v(G)$.

(ii) This is clear from 10.11.

(iii) Use (ii) and 10.8(ii).

10.13. Proposition. $\bar{v}:u = \bar{u}:v = u:v = v:u = w$.

Proof. Let $G \in A$ and let f denote the natural projection of G onto $H = G/u(G)$. We have $w(H) = \bar{v}(H)$ by 10.12(i). Consequently, $(w:u)(G) = (\bar{v}:u)(G)$. However, $w(G) \subseteq (w:u)(G)$, we have proved $w \subseteq \bar{v}:u$, and so $w = \bar{v}:u$. Similarly, $w = \bar{u}:v$.

10.14. Proposition. Let $G \in A$ be left permutable. Then $\widehat{ar}(G) = w(G) = v(G)$.

Proof. Put $H = G/\widehat{ar}(G)$. Then $ar(H) = id_H$. However, $(a.ab)(ab) = a(ab.b) = ab.ab = ab$ and $(ab)(a.ab) = a(ab.ab) = a.ab$ for all $a,b \in H$. Hence $ab = a.ab$.
Further, $ba.ab = a(ba.b) = a(b.ab) = a.ab = ab$ and $ab.ba = ba$. From this, $ab = ba$ and H is a semilattice. The rest is clear.

10.15. Corollary. Let $G \in A$ be left permutable and ideal-free. Then G is $\widehat{ar}$-torsion and right-ideal-free.

10.16. Lemma. Let G be a groupoid containing a subgroupoid H such that H is a semigroup of right zeros, $G = H \cup \{0\}$, $0 \notin H$, $0.H \subseteq H$ and $a0 = 0$ for every $a \in G$. Then $G \in A$ is left permutable and $u(G) \subseteq H \times H \subseteq p(G)$.

Proof. Obviously, G is idempotent. Now, we show that G is medial. For, let $a,b,c,d \in G$. If $a,b,c,d \in H$ then $ab.cd = d = ac.bd$. If $d = 0$ then $ab.cd = d = ac.bd$. If $c = 0$ and $a,b,d \in H$ then $ab.cd = cd = c.bd = ac.bd$. If $a = 0$ and $b,c,d \in H$ then $ab.cd = d = ac.bd$. If $a = 0 = c$ and $b,d \in H$ then $ab.cd = cd = c.bd = ac.bd$. If $a = b = c = 0$ and $d \in H$ then $ab.cd = ac.bd$. Finally, we show that G is left permutable. For, let $a,b,c \in G$. If $a,b,c \in H$ then $a.bc = b.ac$. If $c = 0$ then $a.bc = c = b.ac$. If $a = 0$ and $b,c \in H$ then $a.bc = ac = b.ac$. If $a = 0 = b$ and $c \in H$ then $a.bc = b.ac$.

10.17. Example. Consider the following three-element groupoid $G = \{a,b,c\}$; $aa = ba = ca = a$, $ab = bc = cc = c$, $ac = bb = cb = b$. By 10.16, $G \in A$ and G is left permutable. Moreover, it is easy to see that $p(G) = ar(G) = u(G) = id_G \cup \cup \{(b,c),(c,b)\}$. Hence $u(G) \neq id_G$.

10.18. Lemma. Let n be a non-negative integer and let $G \in A$ be u^n-torsionfree. Then $v(G) = w(G) = \bar{v}(G)$.

Proof. We show by induction on n that $\bar{v}(G) = w(G)$. With respect to 10.12(i), we can assume that $n \geq 2$. Denote by f the natural projection of G onto $H = G/u^{n-1}(G)$ and by g that of G onto $K = G/\bar{v}(G)$. According to 10.4(iii), every block of $u^{n-1}(G)$ is a semigroup of right zeros, and hence $u^{n-1}(G) \subseteq \bar{v}(G)$. Using this, we see that there is a projective homomorphism h of H onto K such that $g = hf$.
Now, let $(a,b) \in w(G)$. Then, by the induction hypothesis, $(f(a),f(b)) \in \bar{v}(H)$, and so $(g(a),g(b)) \in \bar{v}(K)$. Consequently, $(a,b) \in (\bar{v}:\bar{v})(G) = \bar{v}(G)$.

10.19. Proposition. $\bar{v}:u^n = w = \bar{u}:v^n$ for every positive integer n.

Proof. Let $G \in A$ and let f denote the natural projection of G onto $H = G/u^n(G)$. Let $(a,b) \in w(G)$. Then $(f(a),f(b)) \in w(H) = \bar{v}(H)$ by 10.18, and hence $(a,b) \in (\bar{v}:u^n)(G)$.

10.20. Corollary. $v^n:u^m = w = u^m:v^n$ for all positive integers n, m.

11. An application. A congruence r of a groupoid G is said to be e-invariant (resp. a-invariant) if it is invariant with respect to all endomorphisms (resp. automorphisms) of G.

The groupoid G is said to be e-simple (resp. a-simple) if it is non-trivial and id_G, $G\times G$ are the only e-invariant (resp. a-invariant) congruences of G.

11.1. Proposition. Let A be a non-empty abstract class of groupoids and r a semipreradical (resp. a preradical). If $G\in A$ is a-simple (resp. e-simple) then either $r(G) = id_G$ or $r(G) = G\times G$.

11.2. Proposition. Every e-simple distributive groupoid is either idempotent or a semigroup with zero multiplication. Conversely, every non-trivial semigroup with zero multiplication is an e-simple distributive groupoid.

Proof. Let G be an e-simple distributive groupoid. The set I of all idempotents of G is an ideal and it is easy to see that $r = (I\times I)\cup id_G$ is an e-invariant congruence of G. If $r = G\times G$ then I = G and G is idempotent.
Suppose that $r \neq G\times G$. Then $r = id_G$, I contains only one element and G is a semigroup nilpotent of class at most 3. Put K = GG and $s = (K\times K)\cup id_G$. Again, K is an ideal of G and s is an e-invariant congruence. If $s = G\times G$ then G = GG and G is idempotent, a contradiction. Thus $s = id_G$, K contains just one element and G is a semigroup with zero multiplication.

11.3. Corollary. Every a-simple distributive groupoid is either idempotent or a two-element semigroup with zero multiplication.

11.4. Proposition. Let G be an e-simple distributive idempotent groupoid. Then exactly one of the following four cases takes place:
(i) $u(G) = G\times G = v(G)$, G is both left and right-ideal free

and G is cancellative.

(ii) $u(G) = id_G = v(G)$ and G is a semilattice.

(iii) $u(G) = id_G$, $v(G) = G \times G$, G is right-ideal-free and G is left permutable.

(iv) $v(G) = id_G$, $u(G) = G \times G$, G is left-ideal-free and G is right permutable.

Proof. By 11.1, $u(G), v(G) \in \{id_G, G \times G\}$. If $u(G) = id_G = v(G)$ then G is a semilattice by 10.2, 10.4(ii) and its dual. If $u(G) = id_G$ and $v(G) = G \times G$ then G is left permutable by 10.4(ii) and G is clearly right-ideal-free. Suppose that $u(G) = G \times G = v(G)$. Then G is both left and right-ideal-free and G is regular (see [1]). However, the regularity of G implies that $p(G)$ is an e-invariant congruence of G. If $p(G) = G \times G$ then G is a semigroup of right zeros, and, since it is left-ideal-free, it is trivial, a contradiction. We have proved that $p(G) = id_G$, and hence G is right cancellative. Similarly, G is left cancellative.

11.5. Lemma. Every non-trivial semigroup of right zeros is an a-simple distributive idempotent groupoid.

11.6. Lemma. (i) If G is a finite a-simple semilattice then every non-zero element of G is an atom.

(ii) The three-element chain is an e-simple semilattice.

11.7. Proposition. Let G be a finitely generated e-simple distributive groupoid. Then exactly one of the following five cases takes place:

(i) G is a finite semigroup with zero multiplication.

(ii) G is a finite semigroup of left zeros.

(iii) G is a finite semigroup of right zeros.

(iv) G is a finite semilattice.

(v) G is a finite quasigroup.

Proof. With respect to 11.2, we can assume that G is idempotent. Denote by A, B and C the classes of left-zero semigroups, right-zero semigroups and semilattices, resp. Then $m_A(G)$, $m_B(G)$ and $m_C(G)$ are e-invariant congruences of G and we can assume that $m_A(G) = m_B(G) = m_C(G) = G \times G$. Since G is finitely generated, G possesses a non-trivial simple factor-groupoid Q and we see that Q is a finite quasigroup. Denote by V the variety generated by Q. Then V is locally finite and $G \in V$. In particular, G is a finite quasigroup.

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