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A CONSTRUCTION OF THE GLEASON SPACE

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Abstract: A construction of the Gleason space.

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The Gleason space of a compact (Hausdorff) space X is the (unique up to homeomorphism) extremally disconnected compact space G(X) which has an irreducible mapping onto X; see e.g. Comfort and Negrepontis [1] page 57 and for the original construction see Gleaason [3]. The aim of this note is to show an easy and short construction of the Gleason space. This construction was inspired by some ideas of Mioduszewski [4].

Recall, a space X is extremally disconnected if for every open set $U \subset X$, the closure clU is open. A continuous mapping $f:X \xrightarrow{onto} Y$ will be called irreducible if for every closed set $F \subset X$, $\mathrm{cl}f(F) \neq Y$ whenever $F \neq X$. If X is compact, then our definition coincides with the usual one which asserts that f is irreducible if there does not exist any proper closed subset F of X such that f carries F onto Y; see [1] page 55.

Lemma 1. If T is a regular topology on a set X, then there exists a completely regular extremally disconnected topology T^* on X such that $T \subset T^*$ and the identity $i:(X,T^*) \longrightarrow$ $\longrightarrow (X,T)$ is irreducible, where (X,T) and (X,T^*) denote X endowed with the topology T and T^* respectively.

Proof. Let S be the set of all regular topologies on X containing T and such that

(1) if $T' \in S$, then the identity $i:(X,T') \longrightarrow (X,T)$ is irreducible.

The set S is ordered by inclusion. Note that if $L \subset S$ is a chain, then the topology T' generated by $\cup L$ belongs to S. Indeed, T' is regular because $\cup L$ is a base of T' and all topologies in L are regular. To show that $i:(X,T') \longrightarrow (X,T)$ is irreducible, suppose that X - U is dense in (X,T) for some $U \in \cup L$. Since $U \in T''$ for some $T'' \in L \subset S$, we get a contradiction with condition (1). So, every chain in S is bounded. Hence, by the Kuratowski-Zorn Lemma, there exists in S a maximal element T^*. It remains to show that (X,T^*) is extremally disconnected; note that extremally disconnected regular spaces have bases consisting of closed-open sets, hence they are completely regular. Suppose, $U \in T^*$ and $clU \notin T^*$, where cl denotes the closure in T^*. Let T' be the topology generated by $T^* \cup \{clU\}$. Clearly, T' is regular and $T^* \cup \{V \cap clU : V \in T^*\}$ is a base of T'. If $X - (V \cap clU)$ is dense in (X,T), then $V \cap clU = \emptyset$. Indeed, $X - (V \cap U)$ is dense in (X,T) and $U \cap V \in T^* \in S$, hence $U \cap V = \emptyset$. Therefore $T' \in S$, and we get a contradiction with the maximality of T.

<u>Lemma 2</u>. If X is regular, $f: X \xrightarrow{onto} Y$ is continuous and $f|G$ is irreducible for some dense $G \subset X$ such that $f(G) = Y$, then f is irreducible.

Proof. Suppose the contrary. Then $f(X - U)$ is dense in

Y for some open $U \subset X$, $U \neq \emptyset$. Since X is regular, we can assume that U is regularly open, i.e. Int clU = U. We claim that

(2) G - U is a dense subset of X - U.

Indeed, if $V \subset X$ is regularly open and $V \cap (X - U) \cap (G - U) = \emptyset$, then $G \cap V \subset U$. Thus $clV \subset clU$, because G is dense in X. Since V and U are regularly open, $V \cap (X - U) = \emptyset$. Hence, the condition (2) is proved. Now, since f is continuous and f(X - U) is dense in Y, f(G - U) is also dense in Y. But f|G is irreducible, so $G \cap U = \emptyset$; a contradiction.

<u>Construction of the Gleason space</u>: Let X be a compact space. By Lemma 1, there exist a completely regular extremally disconnected space Y and an irreducible mapping f from Y onto X. Clearly, f has a continuous extension $\overline{f}$ over the Čech-Stone compactification βY. By Lemma 2, $\overline{f}$ is irreducible. It is easy to check that βY is extremally disconnected; see e.g. Engelking [2], page 453. Therefore βY is the Gleason space of X.

<u>Remark</u>. By slight modifications, our construction can be extended to some wider classes of spaces. For such generalizations of the Gleason spaces (so called absolutes) see e.g. expository papers by Ponomarev and Šapiro [5] and Woods [6].

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