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Positive solutions of some quasi-linear elliptic problems


Persistent URL: http://dml.cz/dmlcz/106225

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Abstract: In this paper we prove the existence of positive solution \( u \in C^2(\Omega) \) of the quasi-linear elliptic problem
\[
\begin{cases}
- \sum_{i,j} a_{ij}(u(x))D_{ij}u(x) + a_0(u(x))u(x) = g(x,u(x)), & x \in \Omega, \\
u(x) = 0, & x \in \partial \Omega,
\end{cases}
\]
where \( g: \overline{\Omega} \times \mathbb{R}^+ \to \mathbb{R} \) is a sublinear function.

Key words: Quasi-linear elliptic equations, positive solutions, Schauder fixed point theorem.

Classification: 35J65

1. Introduction. In this note we prove the existence of positive solution \( u \in C^2(\Omega) \) of the quasi-linear elliptic problem
\[
\begin{cases}
- \sum_{i,j} a_{ij}(u(x))D_{ij}u(x) + a_0(u(x))u(x) = g(x,u(x)), & x \in \Omega, \\
u(x) = 0, & x \in \partial \Omega,
\end{cases}
\]
where \( g: \overline{\Omega} \times \mathbb{R}^+ \to \mathbb{R} \) is a \( C^1 \)-function satisfying sublinear condition (see Section 4).

The purpose of this paper is to obtain analogous results as for semilinear elliptic problems with sublinear nonlinearity (see e.g. Amann [2]).

The main idea is to use some results from the linear theory of elliptic problems combined with the Schauder fixed point theorem, the continuity of Nemyckij's operator in Hölder

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spaces and the result of Kramer [9]. Boccardo [3] proved the existence of a positive eigenfunction for a class of quasi-linear operators using a similar method but he was working in Sobolev spaces.

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary $\partial \Omega$ and satisfying condition (S) there exists $M > 0$ such that for every pair of points $x, y \in \Omega$ there exist points $x = z_0, z_1, z_2, \ldots, z_n = y$ such that the segments with endpoints $z_i, z_{i+1}$ ($i=0, 1, 2, \ldots, n-1$) are subsets of $\Omega$ and 
\[
\sum_{i=1}^{n-1} |z_i - z_{i+1}| \leq M|x - y|.
\]

**Remark 1.** For details about domains satisfying condition (S) see Kufner, John, Fučík [7]. We need this condition to be true imbedding $c^{k+1}(\overline{\Omega}) \subset c^k(\overline{\Omega})$ (see [7, Thm. 1.2.14]).

We suppose that real functions $a_{ij}, a_0 : \mathbb{R} \to \mathbb{R}$ satisfy the following assumptions:
\[
(2) \quad \begin{cases}
a_{ij}(s) = a_{ji}(s) & \forall s \in \mathbb{R}, \\
\alpha |\xi|^2 \leq \sum a_{ij}(s) \xi_i \xi_j \leq \beta |\xi|^2 & \forall \xi \in \mathbb{R}^N, \forall s \in \mathbb{R}, \\
0 \leq a_0(s) \leq \gamma & \forall s \in \mathbb{R},
\end{cases}
\]
where $\alpha, \beta, \gamma$ are some positive constants.

Moreover let
\[
(3) \quad a_{ij} \in C^2(\mathbb{R}), \quad a_0 \in C^1(\mathbb{R}).
\]

Assume that $g : \overline{\Omega} \times \mathbb{R}^+ \to \mathbb{R}$ is a $C^1$-function. We put $X = \{ u \in C^{2, \kappa}(\overline{\Omega}) : u = 0$ on $\partial \Omega \}$ with the norm of $C^{2, \kappa}(\overline{\Omega})$, $Y = C^{1, \kappa}(\overline{\Omega})$, $Z = C^0(\overline{\Omega})$ (see [7] for usual Hölder space notation).
2. Some auxiliary assertions. The purpose of this section is to prove some auxiliary results which we shall need in the following sections.

Let \( w \in \mathcal{Y} \) be fixed. We shall denote

\[
L(w)v = - \nabla \cdot D_j(a_{i,j}(w(x))D_jv) + a_o(w(x))v.
\]

Put \( a^w_{i,j}(x) = a_{i,j}(w(x)), a^w_o(x) = a_o(w(x)), x \in \overline{\Omega} \). From (2) it follows

\[
a^w_{i,j}(x) = a_{i,j}(x), \quad \forall x \in \Omega,
\]

\[
(\forall \xi) \quad \alpha |\xi|^2 \leq \sum a^w_{i,j}(x) \xi_i \xi_j \leq \beta |\xi|^2, \quad \forall x \in \overline{\Omega},
\]

where the positive constants \( \alpha, \beta, \gamma \) are independent of \( w \in \mathcal{Y} \).

Remark 2. Using assumption (3) and the author's result [4, Thm 1], we obtain that \( a^w_i \in \mathcal{Y}, a^w_o \in \mathcal{Z} \) for all \( w \in \mathcal{Y} \). Hence we are able to apply the Schauder's theory and the \( L^p \)-theory for the Dirichlet problem

\[
(4) \begin{cases}
L(w)u = f \text{ in } \Omega, \\
u = 0 \text{ on } \partial \Omega,
\end{cases}
\]

\( f \in \mathcal{Z} \), for each fixed \( w \in \mathcal{Y} \). Namely, the Dirichlet problem (4) is uniquely solvable and satisfies the a priori estimates:

\[
\|u\|_X \leq c\|f\|_Z,
\]

\[
\|u\|_{W^{2,p}(\Omega)} \leq c\|f\|_{L^p(\Omega)},
\]

where the constant \( c > 0 \) is independent of \( f \in \mathcal{Z} \) and \( w \in \mathcal{Y} \) (see Agmon, Douglis, Nirenberg [1, Thm 7.3, 15.2]).

Remark 3. Let \( w \in \mathcal{Y} \) be fixed. We shall write \( L \) instead of \( L(w) \) in this remark. Let us denote by \( a'_j(m), \) resp. \( a'_j(m), \)
the positive eigenvalues of the eigenvalue problem with an indefinite weight:

(7) \[ \begin{aligned}
    & \begin{cases}
        Lu = m(x)u & \text{in } \Omega, \\
        u = 0 & \text{on } \partial \Omega,
    \end{cases} \\
\end{aligned} \]

resp.

(8) \[ \begin{aligned}
    & \begin{cases}
        -\Delta u = \sigma m(x)u & \text{in } \Omega, \\
        u = 0 & \text{on } \partial \Omega,
    \end{cases} \\
\end{aligned} \]

where \( m \) is a \( C^1 \)-function in \( \Omega \), \( m \neq 0 \). If \( m(x) > 0 \) in \( \Omega_1 \subset \Omega \), \( \text{meas } \Omega_1 > 0 \), it is known (see e.g. de Figueiredo [5, Prop. 1.10]) that (7), resp. (8), has a sequence of such eigenvalues, with a variational characterization. Moreover \( \lambda_1(m) \), resp. \( \sigma_1(m) \), is simple and the corresponding eigenfunctions are of the same sign in \( \Omega \). Lastly \( m < \hat{m} \) in \( \Omega \) implies \( \lambda_j(\hat{m}) < \lambda_j(m) \), resp. \( \sigma_j(\hat{m}) < \sigma_j(m) \), and \( \lambda_j(m) \), resp. \( \sigma_j(m) \), is a continuous function of \( m \) in the norm of \( L^{N/2}(\Theta) \) (see [5, Prop. 1.12A and 1.12B]).

**Lemma 1.** For each \( \omega \in \Theta \) it is

\[ \lambda_1(m) \in [\omega \sigma_1(m), (\beta + \gamma/\sigma_1(1)) \sigma_1(m)]. \]

**Proof.** Let us denote by \( u_1 \), resp. \( v_1 \), the first positive eigenfunction of (7), resp. (8). From the variational characterization of \( \lambda_1(m) \), \( \sigma_1(m) \) and integration by parts we obtain

\[ \lambda_1(m) \int_\Omega m(x)|u_1(x)|^2 \, dx = \int_\Omega Lu_1(x)u_1(x) \, dx \geq \int_\Omega |\nabla u_1(x)|^2 \, dx \geq \omega \sigma_1(m) \int_\Omega m(x)|u_1(x)|^2 \, dx. \]

On the other hand we obtain

\[ \lambda_1(m) \int_\Omega m(x)|v_1(x)|^2 \, dx \leq \int_\Omega Lv_1(x)v_1(x) \, dx \leq \beta \int_\Omega |\nabla v_1(x)|^2 \, dx + \gamma \int_\Omega |v_1(x)|^2 \, dx \leq (\beta + \gamma/\sigma_1(1)) \sigma_1(m) \int_\Omega m(x)|v_1(x)|^2 \, dx. \]

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and the lemma is proved. Q.E.D.

Let \( 0 \leq \mu < \alpha \sigma^{-1}_1(m) \). We are interested in a priori estimates of the solution \( u(w) \in X \) of

\[
L(w)u(w)(x) = \mu m(x)u(w)(x) + f(x), \quad x \in \Omega,
\]
where \( f \in Z \) is given.

**Lemma 2.** There exists a constant \( c > 0 \) independent of \( w \in Y \) and \( f \in Z \) such that

\[
\|u(x)\|_X \leq c \| f \|_Z.
\]

**Proof.** Using Riesz–Fréchet representation theorem it is possible to write the equation (9) in the operator form

\[
u - \mu Tu = f,
\]
where \( T: W^{1,2}_0(\Omega) \to W^{1,2}_0(\Omega) \) is linear symmetric compact operator and \( \mu \) has a positive distance from the spectrum of \( T \) (see Lemma 1). It follows from Taylor [8, Thm 6.40] that

\[
\|u\|_{W^{1,2}_0(\Omega)} \leq \text{const.} \| f \|_{W^{1,2}_0(\Omega)}
\]
with a constant independent of \( w \in Y \) and \( f \in W^{1,2}_0(\Omega) \). Since \( f \) is a representant of \( f \), we obtain

\[
\|u(w)\|_{W^{1,2}_0(\Omega)} \leq \| f \|_{L^2(\Omega)}.
\]
Hence using Sobolev imbedding theorems (see [7]) the right hand side of (9) is in \( L^p(\Omega) \) for some \( p > 2 \). Applying the estimate (5) and imbedding theorems we obtain that the right hand side of (9) is in \( L^{p_1}(\Omega) \) for \( p_1 > p \). Proceeding further we obtain that the right hand side of (9) is in \( Z \). Lastly, applying the estimate (5) and the inequality \( \| f \|_{L^2(\Omega)} \leq \text{const.} \| f \|_Z \) we obtain...
with a constant independent of \( w \in Y \) and \( f \in Z \). Q.E.D.

**Remark 4.** If we denote \( L^{-1} = (L(w) - \mu m)^{-1} : Z \to X \) then \( L^{-1}f = u(w) \) for \( f \) and \( u(w) \) from (9). Lemma 2 tells us that \( \|L^{-1}\| \leq \text{const.} \) with a constant independent of \( w \in Y \), where \( \|L^{-1}\| \) denotes the usual operator norm.

**Lemma 3.** Let

\[
L(w_n)u(w_n)(x) = \mu m(x)u(w_n)(x) + f_n(x) \quad \text{in } Y \quad \text{and } \quad w_n \to w \quad \text{in } Y, \quad f_n \to f \quad \text{in } Z. \quad \text{Then } u(w_n) \to u(w) \quad \text{in } X, \quad \text{for } \quad n \to \infty.
\]

**Proof.** From the assumption (3) and the author's result [4, Thm 2] we obtain

\[
a_{ij}(w_n) \to a_{ij}(w) \quad \text{in } Y, \quad a_0(w_n) \to a_0(w) \quad \text{in } Z.
\]

Hence

\[
\sum_{\alpha \neq 0} a_{ij}(w_n)D_{ij}v \to \sum_{\alpha \neq 0} a_{ij}(w)D_{ij}v \quad \text{in } Y, \quad \quad a_0(w_n)v \to a_0(w)v \quad \text{in } Z
\]

for each \( v \in X \). Consequently

\[
L(w_n)v \to L(w)v \quad \text{in } Z
\]

for each \( v \in X \). Using (14), Remark 4 and denotation \( L_n^{-1} = (L(w_n) - \mu m)^{-1} \) we obtain

\[
\|u(w_n) - u(w)\|_X \leq \|L_n^{-1}f_n - L^{-1}f\|_X \leq \\
\leq \|L_n^{-1}(L_n - L)L^{-1}f\|_X + \|L_n^{-1}(f_n - f)\|_X \leq \\
\leq \text{const.}(\|L_n(L^{-1}f) - L(L^{-1}f)\|_Z + \|f_n - f\|_Z) \to 0.
\]

Q.E.D.

**Remark 5.** There is proved in [4, Thm 2] that a neces-
ecessary and sufficient condition for the continuity of Nemyckij's operator $a_{ij}(\cdot ) : Y \rightarrow Y$, resp. $a_0(\cdot ) : Z \rightarrow Z$, is (3). This is the reason why using this method of the proof there is not possible to weaken the condition (3).

Let $m \in C^1(\overline{\Omega})$ be the weight function satisfying the assumptions stated in Remark 3. We are ready, now, to prove the following useful assertion.

**Lemma 4.** Suppose that $\mu_1(m) > 1$ for all $w \in Y$, $f \in Z$, $f > 0$ in $\Omega$. Then the problem

$$
\begin{align*}
L(v)v &= m(x)v + f \text{ in } \Omega, \\
v &= 0 \text{ on } \partial \Omega
\end{align*}
$$

has the solution $v \in X$ such that $v > 0$ in $\Omega$ and outward normal derivative $\frac{\partial v}{\partial \nu} < 0$ on $\partial \Omega$.

**Proof.** According to [5, Thm 1.14, 1.17], for each fixed $w \in Y$ there exists the unique solution $v(w) \in X$ of the linear problem

$$
\begin{align*}
L(w)v(w) &= m(x)v(w) + f \text{ in } \Omega, \\
v(w) &= 0 \text{ on } \partial \Omega
\end{align*}
$$

such that $v(w) > 0$ in $\Omega$ and $\frac{\partial v(w)}{\partial \nu} < 0$ on $\partial \Omega$. We shall define the operator $S : Y \rightarrow X$ by the way $S(w) = v(w)$, where $v(w)$ is the unique solution of (15).

Let us suppose that $w_n \rightarrow w$ in $Y$. Applying Lemma 3 we obtain $v(w_n) \rightarrow v(w)$ in $X$. This means that $S$ is continuous from $Y$ into $X$. According to [7, Thm 1.2.14, 1.5.10] we have the compact imbedding $X \subseteq Y$ and hence the restriction $\overline{S} : S|X : X \rightarrow X$ is completely continuous operator. Applying Lemma 2 we obtain the existence of a sufficiently large ball in $X$ centred at the
origin which is mapped by $S$ into itself. Schauder fixed point theorem implies the existence of at least one $v \in X$ such that $S(v) = v$, i.e. $v$ is the solution of (15). Since $v$ is also the solution of (15') with $w = v$ it is $v > 0$ in $\Omega$, $\frac{\partial v}{\partial n} < 0$ on $\partial \Omega$. Q.E.D.

The following result is due to Boccardo [3, Thm 1].

**Lemma 5.** For each positive real number $r$, we can find a positive eigenvalue $\mu$ with the corresponding positive eigenfunction $u \in X$ such that
\[
\begin{cases}
L(u)u = \mu u \text{ in } \Omega, \\
u = 0 \text{ on } \partial \Omega
\end{cases}
\]
and $\|u\|_{L^2(\Omega)} = r$.

**Remark 6.** More precisely, by a direct application of [3, Thm 1] we obtain a positive eigenfunction $u \in Z$. But under our assumptions on the coefficients of the differential operator $L$ Remark 2 immediately implies that $u \in X$.

The following assertion will be very important in the proof of our main existence theorem.

**Lemma 6.** There exists a constant $k > 0$ (independent of $u \in X$ and $r > 0$) such that
\[
\|u\|_X \leq kr,
\]
where $u \in X$, $\|u\|_{L^2(\Omega)} = r$ is the solution of the eigenvalue problem (16).

Proof of this lemma is based on the bootstrap argument used in the proof of Lemma 2 and the uniform estimates (5) and (6) play the key role in proving this assertion.
3. Subsolution, supersolution and the existence of the solution

Definition. A function \( u \in C^2(\Omega) \) is said to be a supersolution of (1) if
\[
L(u)u \geq g(x,u) \text{ in } \Omega, \\
u \geq 0 \text{ on } \partial \Omega.
\]
A function \( u \in C^2(\Omega) \) is said to be a subsolution of (1) if
\[
L(u)u \leq g(x,u) \text{ in } \Omega, \\
u \leq 0 \text{ on } \partial \Omega.
\]

Let us formulate, now, the assertion which is proved in more general setting in Kramer [9].

Lemma 7. Suppose \( u \leq \bar{u} \) (in \( \Omega \)) are sub- and super-solutions of (1). Then there exists at least one solution \( u(x) \in C^2(\Omega) \) of (1) satisfying
\[
u(x) \leq u(x) \leq \bar{u}(x) \text{ in } \Omega.
\]

Remark 7. The result of Kramer [9] is the generalization of the well-known result of Kazdan and Warner for semilinear elliptic problems (see e.g. Fučík [6]).

4. Existence of positive solutions. In this section we shall prove the existence of a positive solution for quasilinear elliptic problem (1) with sublinear nonlinearity \( g(x,s) \).

Let the function \( g \) satisfy the following conditions:

(17) There are constants \( g_0 > 0, s_0 > 0 \) such that
\[
g(x,s) \geq g_0 s \quad \forall x \in \bar{\Omega}, \forall 0 < s < s_0.
\]

(18) There are continuous functions \( g_0, c: \bar{\Omega} \to \mathbb{R} \), with \( c(x) \geq 0 \) such that
Theorem 1. Suppose that the function \( g \) satisfies (17) and (18). Let
\[
\sigma_1'(g_0) < \frac{1}{\beta + \gamma / \sigma_1'(1)},
\]
(19)
and
\[
\sigma_1'(g_\infty) > \frac{1}{\infty}.
\]
(20)
Then the Dirichlet problem (1) has a positive solution.

Remark 8. An analogous theorem for semilinear elliptic problems was firstly proved by Amann [2].

Proof of Theorem 1. Choose the \( C^1 \)-functions \( \hat{g}_\infty, \hat{c} : \Omega \to \mathbb{R} \) such that \( \hat{c}(x) > 0 \),
\[
g(x,s) \leq \hat{g}_\infty(x)s + \hat{c}(x) \quad \forall x \in \Omega, \quad \forall s \geq 0,
\]
(21)
\[
\hat{g}_\infty(x_0) > 0 \quad \text{for some } x_0 \in \Omega \quad \text{and}
\]
\[
\| \hat{g}_\infty - \hat{g}_\infty \|_{L^2(\Omega)} < \varepsilon
\]
for such small \( \varepsilon > 0 \) that the continuous dependence of \( \sigma_1'(m) \) on the weight function \( m \) (see Remark 3) would imply \( \sigma_1'(\hat{g}_\infty) > \frac{1}{\infty} \). According to Lemma 1 it is \( \langle \mu_1(\hat{g}_\infty) \rangle > 1 \) for all \( w \in Y \).

Hence using Lemma 4 the problem
\[
\begin{cases}
L(u)u = \hat{g}_\infty u + \hat{c} \text{ in } \Omega, \\
u = 0 \text{ on } \partial \Omega
\end{cases}
\]
(22)
has the solution \( \hat{u} \in X \) and \( \hat{u} > 0 \) in \( \Omega \), outward normal derivative \( \frac{\partial \hat{u}}{\partial n} < 0 \) on \( \partial \Omega \). Hence the expressions (21) and (22) show that \( \hat{u} \) is a supersolution of (1).

The assumption (19) implies that \( \mu_1(g_0) < 1 \) for all \( w \in Y \). Then according to Lemma 5 the eigenvalue problem
(23) \[
\begin{cases}
L(u)u = \mu g_0 u & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega
\end{cases}
\]

has a positive eigenfunction \( u \in X \) corresponding to the eigenvalue \( \mu_1 < 1 \) and \( \|u\|_{L^2(\Omega)} = r \). According to Lemma 6 the number \( r > 0 \) can be chosen such small that \( u < s_0 \) and \( u < \tilde{u} \) in \( \Omega \).

Then using (17) we obtain

\[
L(u)u = (\mu_1 g_0, u < g(x, y))
\]

which shows that \( u \) is a subsolution of (15). There are fulfilled all the assumptions of Lemma 7 and there exists a solution \( u \in X \) of the problem (1). Note that this solution is such that \( u(x) \geq y(x) > 0 \) for all \( x \in \Omega \). Q.E.D.

Remark 9. Consider the eigenvalue problem

(24) \[
\begin{cases}
- \nabla \cdot D_1 e_{ij}(u(x)) D_1 u(x) + a_0 u(x) u(x) = \lambda f(x, u(x)), \\
u(x) = 0, \quad x \in \partial \Omega
\end{cases}
\]

where \( f: \overline{\Omega} \times [0, +\infty) \to R \) is a \( \mathcal{C}^1 \)-function, and let us suppose that

\[
f_0(x) = \liminf_{s \to \partial \Omega^+} \frac{f(x, s)}{s}, \quad f_\infty(x) = \limsup_{s \to +\infty} \frac{f(x, s)}{s}
\]

are continuous functions. Then if

(i) \( f_0(x) \equiv +\infty \) (in particular if \( f(x, 0) > 0 \)) and \( f_\infty(x) \equiv 0 \), the problem (24) has a positive solution for all \( \lambda > 0 \);

(ii) \( f_0(x) \equiv +\infty \) and \( f_\infty(x_0) > 0 \) for some point \( x_0 \in \Omega \), the problem (24) has a positive solution for all

\[
0 < \lambda < \frac{\max_{x \in \Omega} f_\infty(x)}{f_\infty(x_0)}
\]

(iii) \( 0 < \varepsilon \leq f_0(x) < +\infty \) in \( \overline{\Omega} \) and \( f_\infty(x) \equiv 0 \), the problem (24) has a positive solution for all

\[
\lambda > \frac{\beta \sigma_1(1) + \gamma}{\inf_{x \in \overline{\Omega}} f_0(x)}
\]

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(iv) $0 < x \leq f_0(x) < +\infty$ in $\overline{\Omega}$ and $f_\infty(x_0) > 0$ for some $x_0 \in \Omega$, the problem (24) has a solution for all $\lambda$ such that
$$\frac{\beta \omega'(1) + \gamma}{\inf_{x \in \Omega} f_0(x)} < \lambda < \frac{\omega'(1)}{\sup_{x \in \Omega} f_\infty(x)}.$$

The proof of (i) - (iv) follows immediately from Theorem 1.

References


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(Oblatum 20.10. 1982)