UNIFORM WEIGHT OF UNIFORM QUOTIENTS

MIROSLAV HUŠEK, JAN PELANT

Abstract: Uniform weight of uniform quotients is estimated and it is shown that the estimation cannot be improved. In particular, examples of nonmetrizable uniform quotients of metric spaces are given.

Key-words: uniform space, quotient, uniform weight, metric space

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The uniform weight of a uniform space is the smallest cardinality of a base for uniform covers or of a base for uniform vicinities of diagonal. We shall look how the uniform weight behaves by uniform quotients. This question reduces to investigation of quotients of metric spaces. Some cases when a uniform quotient of a metric space is metrizable as a uniform space are treated e.g. in [C], [H1], [M]. An example that a uniform quotient of a metric space is not pseudometrizable is given in [M], however, we were not able to check all the details. The similar examples presented in this paper are simple and the spaces used have additional nice properties: in the first example, the local character of the quotient space is uncountable and the domain space is discrete, in the second one the quotient map is at most 2 to 1 and the uniform quotient is also a topological quotient, hence, it is metrizable as a topological space.

Ordinal number is understood here as the set of smaller ordinals, initial ordinals are cardinals (thus \(\eta+1\) is the set \(\{0,1,\ldots,\eta\}\), but because of better understanding we shall denote that set by \(\hat{\eta}\)). By \(^A B\) we denote the set of all mappings on \(A\) into \(B\).
Thus \( U \) is the set of all mappings on \( \omega \) into \( \omega \) and we shall endowed it with the pointwise order: \( f < g \) if \( f \upharpoonright n g \) for all \( n \in \omega \).

The \( \text{cof}(\omega) \) is the smallest cardinality of a cofinal set in \( \omega \) and it is consistent with ZFC that \( \text{cof}(\omega) \) equals to any cardinal which is not greater than \( 2^{\omega} \) and has uncountable cofinality, \( [\text{He}] \).

Uniform spaces are given by means of the set of uniform covers, and if \( u \) is a uniformity then \( E(u) \) denotes the corresponding set of uniform vicinities of the diagonal in \( X \times X \). A pseudometric \( d \) on \( X \) is called uniformly continuous on \( (X,u) \) if the uniformity induced by \( d \) is smaller than \( u \) (i.e. \( d \) is a uniformly continuous function on \( (X,u) \)).

In the sequel, \( q: (X,u) \twoheadrightarrow (X,v) \) is a uniform quotient mapping between uniform spaces, i.e. \( v \) is the biggest uniformity on \( Y \) making \( q: (X,u) \twoheadrightarrow (X,v) \) uniformly continuous. The uniformity \( v \) may be described by means of uniformly continuous pseudometrics \( d \) on \( (Y,v) \) (\( d \upharpoonright (X,u) \) is uniformly continuous on \( (X,u) \)), or as the set of covers of \( Y \), initiating a normal sequence in the image \( q(x) \). We shall describe the quotient in a way more convenient for our purposes, using the technique described e.g. in [DR].

For \( r > 0 \) and a uniformly continuous pseudometric \( d \) on \( (X,u) \) we denote \( M_d(r) = \{(a,b) \in X \times X \mid d(a,b) < 1/r \} \); if \( f \) is an increasing \( \omega \)-mapping \( \omega \rightarrow \omega \) (0), then \( M_d(f) = \{ M_d(f(p0)) \cup M_d(f(p1)) \cup \ldots M_d(f(p \omega)) \} \) \( \omega \)-uniformly continuous on \( U \). Thus \( M_d(f) \) is a uniformly continuous pseudometric on \( (X,u) \) such that \( d(a,b) < 1 \), and \( f \) is an increasing map on \( \omega \) into \( \omega \) such that \( M_d(f) \) is \( \omega \)-uniformly continuous on \( (X,u) \) such that \( d(a,b) < 1 \), and \( f \) is an increasing map on \( \omega \) into \( \omega \) such that \( M_d(f) \) is \( \omega \)-uniformly continuous on \( (X,u) \) such that \( d(a,b) < 1 \). It remains to show that the collection \( \{ M_d(f) \} \) is a base for a uniformity; the only non-trivial part is to show that for each \( d,f \) there are \( e,g \) such that \( M_d(f) \supseteq M_e(g) \supseteq M_e(g) \). To do that, it suffices to put \( e=d \upharpoonright n = f(2\omega+2) \), \( M_d(g(p0)) \cup M_d(g(p1)) \cup \ldots M_d(g(pn)) \cup M_d(f(2p0+2) \ldots M_d(f(pn)) \cup M_d(f(2p0+2) \ldots
COROLLARY 1. If the uniform weight of \((X, t_u)\) is \(\kappa\), then the uniform weight of its quotient \((Y, t_v)\) is less or equal to \(\kappa \cdot \text{cof}(\omega)\).

Proof. Clearly, if \(f < g\) then \(M_d(f) \supseteq M_d(g)\). If \(e\) is a uniformly continuous pseudometric on a pseudometric space \((X, d)\), then \(M_e(f) \supseteq M_d(g)\) for some convenient \(g\) (since for each \(n\) there exists \(m\) such that \(d(a, b) < 1/m\) implies \(e(a, b) < 1/n\)).

COROLLARY 2. Assume that \(X\) is a uniform space with uniform weight, not smaller than \(\text{cof}(\omega)\). If \(X\) has a monotone base, then any uniform quotient of \(X\) has a monotone base, too.

Proof. A uniform space is said to admit cardinal \(\kappa\) if \(\kappa\)-many uniform covers have a common uniform refinement. A uniform space \(X\) has a monotone base iff \(X\) admits any cardinal smaller than its uniform weight. Since every quotient of \(X\) admits the cardinals admitted by \(X\), our assertion follows from Corollary 1.

In fact, we have proved more, namely that if \(X\) is the space from Corollary 2, then its uniform quotient is either uniformly discrete or has the same uniform weight as \(X\).

We shall show now that the estimation given in Corollary 1 of the uniform weight of \((Y, t_v)\) cannot be improved, i.e. that a uniform quotient of a metric space has uniform weight equal to \(\text{cof}(\omega)\).

EXAMPLE 1. There is a complete countable metric space \(X\) which is topologically discrete and has a uniform quotient \(Y\) such that every point of \(Y\) has local character equal to \(\text{cof}(\omega)\).

Denote \(Y = (\beta)(\mathbb{R}((\omega-0))|n \in \omega), X = X, Y, q\) the projection \(X\) onto \(X\).

The metric \(d\) on \(X\) is defined as follows (by \(x \sim y \Leftrightarrow \) for \(y \in \mathbb{R}(\omega-0)\) we describe the situation when \(s \in \mathbb{R}(\omega-0), s\) extends \(x, y\) and \(s(k+1) = n, y \sim s \Leftrightarrow \) we mean \(s \in \mathbb{R}(\omega-0), s(0) = n)\)

\[
d((x_1, y_1), (x_2, y_2)) = \begin{cases} d((x_1, y_1), (x_2, y_2))^{1/\alpha} if \ y_2 = y_1 \Rightarrow n, x_2 = 0, x_1 = 1; \\
1 otherwise.
\end{cases}
\]

The metric \(d\) is complete and induces the discrete topology on \(X\). On \(Y\), we take the quotient uniformity along \(q: (X, d)\). For \(y \in Y\) and \(n \in (\omega-0), N(n)(y) = \{x \in X | e^k(x, y) \leq \text{cof}(\omega)\} = (n)\) for each \(n \in \mathbb{R}(\omega-0), n \geq 0\) and \(k > n\) hence \(M(f)(\beta) = \{x \in X | e^k(x, y) \leq \text{cof}(\omega)\} = (n)\) for each \(n \in \mathbb{R}(\omega-0), n \geq 0\) and \(k > n\).
Suppose now that \( \{M(f) \mid f \in F \} \) is a local base at \( \emptyset \) in \( Y \) and \(|F| < \text{cof}(w)\), then there is \( g \in \text{cof}(w-0) \) which is not bounded from above by any \( f \in F \). We can find \( f \in F \) such that \( N(g)(\emptyset) \supseteq N(f)(\emptyset) \) and \( \pi \in \omega \) with \( gn > fn \). Take now such a \( \pi \), \( f \in F \) such that \( s(k) = fk + 1 \) for all \( k \in \omega \); then \( s(M(f)) \subseteq N(g)(\emptyset) \), which is a contradiction. Indeed, if \( s \in N(g)(\emptyset) \), then there is \( \{u_i\}_{i=0}^{k} \subseteq \pi \) such that \( u_0 = \emptyset, u_k = z \) and \( (u_i, u_{i+1}) \in N(g(p_i)) \) for \( i < k \). For each \( j \in \omega \), there exists \( i < k \) such that \( u_{i+1} = u_i \) and \( s(j) \), hence there exists an injection \( \phi : \pi \to \pi \) with \( \phi \) which is not possible. The same procedure works for other points \( y \in Y \).

The map \( q \) from Example 1 cannot be expected to be finite-to-one and the space \( Y \) cannot be the topological quotient of \( X \). We shall now construct another example, where the map \( q \) is at most \( 2 \) to \( 1 \) and is also the topological quotient, but the cardinality of \( X \) is uncountable. It follows from one result of Arens-Gelshnikij in [A] that the quotient space \( Y \) is metrizable as a topological space.

**Example 2.** There is at most \( 2 \) to \( 1 \) mapping \( q \) defined on a Baire space \( D \) such that uniform weight of the quotient along \( q \) is \( \text{cof}(w) \). The quotient space is topologically metrizable.

Let \( D \) be a cofinal set in \( \omega^\omega(\omega-0) \) endowed with the uniformly discrete uniformity and \( X = D^\omega(\omega-0) \) be endowed with the Baire metric \( d(\{x_i\}, \{y_i\}) = 1/n \), where \( n \) is the first coordinate with \( x_n \neq y_n \).

Choose a countable subset \( \{s_n\} \) in \( D \) and for every \( f \in D, n \in \omega \), define \( a_f^n, b_f^n : X \to \pi \)

\[
a_f^n(i) = \begin{cases} f & \text{if } i = 1 \\ 2^n & \text{if } i > 1 \end{cases} \quad \quad b_f^n(i) = \begin{cases} f & \text{if } i = 1 \\ 2^n & \text{if } 1 < i \leq fn \\ 2^n & \text{if } i > fn \end{cases}
\]

The quotient map \( q : X \to Y \) is defined by means of the equivalence: \( qa = qb \) if either \( a = b \) or there is \( f \in D, n \in \omega \) such that either \( a = a_f^n \), \( b = b_f^n \). Since \( N(n) = \{(a,b) : (a,b) \in X \times Y \} \) there are \( x \neq y \) with \( qx = ax, qy = by, x_i = y_i \) for all \( i \in n \), the pair \( \{qa^n, qa^n\} \) always belongs to \( M(f) \).

Suppose that the uniform quotient \( Y \) of \( X \) has a base \( \{N(f) \mid f \in F \} \) of cardinality less than \( \text{cof}(w) \) and take \( g \in \omega^\omega \) such that \( g < f \).
We may suppose that \( g_0 \) and all \( f_0 \) for \( f \in F \) are bigger than 1. There is some \( f \in F \) such that \( M(g) > M(f) \) for all \( g \in \mathbb{R} \). We shall show that \( (q_{a_f}, q_{a_{f+1}}) \notin M(g) \), which contradicts the previous facts. If \( (q_{a_f}, q_{a_{f+1}}) \in M(g) \), then there are points \( u_i \in X \) for \( i < k \) and a permutation \( p \) on \( k \) such that \( u_0 = a_f^0, u_k = a_{f+1}, (u_{i-1}, u_{i+1}) \in M(g(p_i)) \) for all \( i < k \). Since \( g_0 > 1 \), for every \( i \leq n + 1 \) there is \( \psi \) such that \( u_i \psi = q_{a_f^i} \) and the mapping \( \psi : n+1 \to k \) preserves ordering; moreover, for every \( i \leq n \) there must be a \( \psi \) such that \( \psi \leq q_{a_f^i} \psi (i+1) \) and \( g \psi \leq f \) (since \( (q_{a_f^i} q_{a_{f+1}^i}) \in M(g(p(i+1))) \) but that is impossible because there is at most \( n-1 \) points in \( k \) in which \( g \) has value less or equal to \( f_n \).

At the end we would like to add a remark concerning the behaviour of uniform pseudoweight by quotients. Similarly as pseudocharacter in topological spaces, uniform pseudoweight of a uniform space \( (X, \mathcal{U}) \) is the least cardinality \( \kappa \) for which there exists \( \nu \in \mathcal{U} \) with \( |\nu| = \kappa \) and such that the meet of \( \nu \) coincides with that of \( \nu \) for separated spaces it means that \( n(\nu) \) is the diagonal. We shall now provide an example showing that there is no simple connection between uniform pseudoweights of a space and its quotient.

**Example 3.** For each cardinal \( \kappa \) there is a uniform quotient \( q : X \to Y \) such that \( X \) has countable uniform pseudocharacter and uniform pseudocharacter of \( Y \) is not smaller than \( \kappa \).

Let \( \kappa \) be an infinite regular cardinal and \( Y \) be the uniform space with the underlying set \( \kappa \times 2 \) and with the base of uniform covers
\[
\{(y) | y \in Y\} \cup \{(a, 0), (a, 1) | a > \beta \} \quad \text{for} \ \beta \in \kappa.
\]
Uniform pseudoweight of \( Y \) is \( \kappa \). We shall show that \( Y \) is a uniform quotient of a space having countable pseudoweight. For each cofinal set \( S \) in \( \kappa \) we may find a monotone sequence \( \{S_n \} \) such that each \( S_n \) is cofinal in \( S, n S_n = \emptyset \). Let \( X_S \) be the uniform space with the same underlying set as \( Y \) has and with the base of uniform covers
\[
\{(y) | y \in Y\} \cup \{(a, 0), (a, 1) | a \in S_n, a > \beta \} \quad \text{for} \ \beta \in \kappa, n \in \omega.
\]
Uniform pseudoweight of \( X_S \) is \( \omega \), and the uniformity of \( Y \) is the biggest uniformity contained in the uniformities of the above spaces \( X_S \). Thus \( Y \) is a uniform quotient of the sum of spaces \( X_S \).
REFERENCES


Mathematical Institute
Charles University
Sokolovská 83
18600 Prague
Czechoslovakia

Czechoslovak Academy of Sciences
Žitná 25
11567 Prague
Czechoslovakia

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