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A possible modal reformulation of comprehension scheme

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# ANNOUNCEMENTS OF NEW RESULTS

## SOME THEOREMS ON THE LATTICE OF LOCAL INTERPRETABILITY TYPES

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In [2], J. Mycielski introduced the lattice of multidimensional local interpretability types of theories (for next shortly type) and he posed some problems (see also a joint manuscript [1] with A. Ehrenfeucht). By next three theorems we solved two of them. $|T|$ denotes type of a theory T and $\leq$ denotes ordering in the lattice.

Theorem 1: A type is meet-irreducible iff it contains a complete theory.

Corollary of the proof: If S is a finitely axiomatizable and essentially undecidable theory and R is its recursively axiomatizable extension then type $|R|$ is not meet-irreducible.

Theorem 2: For each two types t, s such that $s \not\leq t$ there exists a meet-irreducible type $t' \geq t$ such that still $s \not\leq t'$.

Corollary: For each type t which is not maximal there exists a meet-irreducible type $t' \geq t$ which is still not maximal.

Theorem 3: Each type contains a theory with a finite language.

Also some results about mutual multidimensional interpretability of various theories of order are proved. Let us define six theories in the language $\{=; \prec\}$ (= is equality and in the following theories we assume implicitly the axioms of equality, $x \preceq y$ stands for $x \prec y \vee x = y$):

PO: $\forall x; \neg x \prec x + \forall x,y,z; (x \prec y \,\&\, y \prec z) \longrightarrow x \prec z + \forall x \exists y; x \prec y$

POD: $PO + \forall x \forall y \succ x \exists t \succ x; y \succ t + \forall x \exists y; y \prec x$

POS: $PO + \exists! x \forall y; x \preceq y + \forall x \forall y \succ x \exists z \succ x \forall t \succ x; z \preceq y \,\&\, (t \preceq y \rightarrow z \preceq t)$
$+ \forall x \forall y \prec x \exists z \prec x \forall t \prec x; y \preceq z \,\&\, (y \preceq t \rightarrow t \preceq z)$

LO: $PO + \forall x,y; x \preceq y \vee y \preceq x$; LOD: POD + LO; LOS: POS + LO

In the following theorem $\leq_1$ denotes 1-dimensional interpretability and $\perp$ denotes incomparability with respect to multidimensional (!) interpretability.

Theorem 4: (i) $PO < LO, POS, POD$ (ii) $LO < LOS, LOD$
(iii) $POS < LOS$ (iv) $POS \perp POD, LO, LOD$
(v) $LOS \perp POD, LOD$ (vi) $|LO| = |LOS| \wedge |LOD|$
(vii) for each theory T: $LO \leq T \longrightarrow LO \leq_1 T$.

References: [1] A. Ehrenfeucht, J. Mycielski: Theorems and problems on the lattice of local interpretability, manuscript

[2] J. Mycielski: A lattice of interpretability types of theories, Journal of Symbolic Logic 42, 1977

## A POSSIBLE MODAL REFORMULATION OF COMPREHENSION SCHEME

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In various set theories the Cantor's comprehension is reformulated (Quine's NF) or replaced by a set of axioms (ZF, GB).

We consider a possible reformulation of this principle using modal logic.
Our main idea is that it seems to us (from point of view of our knowledge) that set universum behaves as if Cantor's comprehension were sound. More formally: we use the modal lower predicate calculus with identity (we adopt the rule of necessitation too) as is formalized in [1] and we prefer to think about " $\Box\varphi$ " as "we know $\varphi$" (i.e., epistemic modality).
Additional modal axioms are those of system T (see [1]); $\Box\varphi \longrightarrow \varphi$ and $\Box(\varphi \longrightarrow \psi) \longrightarrow (\Box\varphi \longrightarrow \Box\psi)$. The formal theory MST consists (except preceding logical axioms) of these principles:

(i) $(\forall t;\, t \in x \equiv t \in y) \longrightarrow x = y$ extensionality

(ii) $\Diamond x = y \longrightarrow \Box x = y$

(iii) for any formula $\varphi(t)$ (possibly with modality and parameters) holds:

$$\exists y\, \forall t;(\Box\varphi(t) \equiv \Box\, t \in y)\ \&\ (\Box\neg\varphi(t) \equiv \Box\, t \notin y)$$

So the last principle is formalization of our main idea.
Theory MST interprets, for example, arithmetic with bounded induction $I\Delta_0$ thus it contains some nontrivial mathematics.
It also proves existence of an infinite set.
But much more can be done. By strengthening of underlying logic (namely by S4, Brouwer's axiom and Barcan's formula - see [1]) we are able to develop Peano arithmetic with full induction.
We are not able to prove consistency of our system. About this problem we have only results of partial relative consistency with respect to ZF (in fact to Gilmore's PST) and to Quine's NF. On the other hand we know that some strong modal axioms (namely S5) make the theory contradictory.

Reference: [1] G.E. Hughes, M.J. Cresswell: An introduction to modal logic, Methuen and Co. Ltd., 1968.

## CONCERNING THE FINE TOPOLOGY FROM THE BAND $\mathcal{M}$

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The coarsest topology m making all potentials from the band $\mathcal{M}$ continuous does not generally satisfy the following "quasilindelöf property":

Every family of m-open sets contains a countable subfamily whose union differs from the union of the whole family by a polar (or negligible) set.

The heat equation on $R\times R$ with the cover $\{U_y\}_{y\in R}$ of $R^2$ where $U_y = \{(x,t): t\neq 0 \text{ or } x=y\}$ serves as a counterexample. The sets $U_y$ are m-open by the following

Theorem. Let X be a $\mathfrak{P}$-harmonic space with a countable base ([1]). If $A\subset X$ is thin at a polar point $z\in X\setminus A$ and $\bar{A}\subset A\cup\{z\}$, then $X\setminus A$ is m-open.

Proof. Let v be a potential on X which is $+\infty$ at z ([1], Ex. 6.2.1) and p be a strictly positive finite continuous potential on X. From [1], Ex. 8.2.2 and C.5.3.2 we deduce $\inf\{R_p^{A\cap V}(z):\ V \text{ is a neighborhood of } z\} = 0$. Hence there is a