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Topological categories with both epireflective and coreflective proper subcategories

Commentationes Mathematicae Universitatis Carolinae, Vol. 24 (1983), No. 3, 481--488

Persistent URL: http://dml.cz/dmlcz/106247

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Abstract: It is known that some familiar topological categories - for example the category of all topological spaces and the category of all uniform spaces - have no proper subcategories which are both epireflective and coreflective. In this paper we produce a class of topological categories which contain the category Re of all reflexive relations as proper, both epireflective and coreflective subcategory. All these categories are cartesian closed and have other nice properties.

Key words: Topological category, epireflective subcategory, coreflective subcategory.

Classification: 18A40

1. It is shown in ([6]) that the topological category $\text{Top}$ of all topological spaces has no proper subcategory which is both epireflective and coreflective (i.e. closed under the formation of products, subspaces, coproducts and quotient spaces). The same statement is true for the categories $\text{Mer}$ of all merotopic spaces ([7]), $\text{Unif}$ of all uniform spaces ([5]) and $\text{Born}$ of all bornological spaces ([11]).

The category $\text{Simp}$ of all abstract simplicial complexes ([91]) is an example of a topological category with a proper subcategory which is both epireflective and coreflective. In fact the subcategory $\text{Simp}_0$ of all o-dimensional simplicial
complexes (which coincides with the subcategory $\text{Discrete(Simp)}$ of all discrete Simp-objects) is obviously closed under the formation of products, subspaces, coproducts and quotient spaces. Moreover it is shown in [1] that $\text{Simp}_0$ is the unique subcategory of $\text{Simp}$ with the previous properties.

Examples of topological categories (in the sense of Herrlich [2]) with a proper, non trivial (i.e. $\neq \text{Discrete}$) both epireflective and coreflective subcategory were firstly given by Hušek in [5] and [10].

Example 1 ([5] Example 2). Let $\mathcal{X}$ be the full subcategory of the category $\text{Unif}$ whose objects are all uniform spaces $(X, U)$, $U$ uniform neighborhoods of diagonal, with the property that the intersection of equivalences from $U$ belongs to $\mathcal{U}$. If $\mathcal{X}$ is the full subcategory of $\mathcal{X}$ generated by all $(X, U)$ such that $\cap U \in \mathcal{U}$ then $\mathcal{X}$ is non trivial, and both coreflective and bireflective in $\mathcal{X}$.

Example 2 ([10] Example 6). Let $\mathcal{X}$ be the full subcategory of the category $\text{Top}$ whose objects are all locally connected spaces. Then $\mathcal{X}$ is a topological category because it is a coreflective subcategory of the topological category $\text{Top}$. The full subcategory $\mathcal{Y}$ of $\mathcal{X}$ composed of all spaces the collection of open and closed sets of which coincides is a non trivial, both bireflective and coreflective subcategory of $\mathcal{X}$.

The aim of this paper is to describe a class of topological categories which have proper, non trivial, both epireflective and coreflective subcategories. In particular each of these categories contains as full subcategory with the previous properties the category $\text{Rele}$ of all reflexive relations.
described in [3]. It is also shown that all these categories are strongly topological (i.e. they are cartesian closed and final epi-sinks are hereditary) and no universal topological categories in the sense of Marny ([8]).

2. Here topological category means concrete category \((X,U)\) over the category \(\text{Set}\) of all sets which is initially complete and well-fibred (i.e. it is small-fibred and every constant map \(UX \to UY\) underlies some \(X\)-morphism).

For definitions and results on cartesian closed topological categories see [2]. All undefined terminology is that of [4].

2.1. Definition. Let \(A\) be a non-empty set. \(A\) denotes the category whose objects are couples \((X,F)\) where \(X\) is a set and \(F\) is a subset of the set \(\text{Set}(A,X)\) of all maps of \(A\) into \(X\) containing all constant maps, and whose morphisms of \((X,F)\) into \((Y,G)\) are the maps \(h:X \to Y\) such that \(h \circ f \in G\) for each \(f \in F\).

2.2. Proposition. \(A\) is a topological category.

Proof. The class of all \(A\)-structures in a set \(X\) is a set and every constant map is an \(A\)-morphism. Thus \(A\) is well-fibred. Let \(X\) be a set, \((Y_i,F_i)_{i \in I}\) a family of \(A\)-objects and \((h_i:X \to Y_i)_{i \in I}\) a family of maps. Then the set \(F = \{f:A \to X: (h_i \circ f) \in F_i\ \text{for each } i \in I\}\) is the initial \(A\)-structure in \(X\) with respect to \((X,(h_i,F_i))_{i \in I}\) uniquely determined by its defining properties. Thus \(A\) is initially complete.

2.3 (1) An epi-sink \((h_i:(X_i,F_i) \to (Y,F))_{i \in I}\) is final iff \(f \in F\) holds whenever there exist \(i \in I\) and \(f_1 \in F_i\) such that \(h_i \circ f_1 = f\).
(2) \( k: (X, F) \rightarrow (Y, G) \) is an embedding (and \((X, F) \) is a subspace of \((Y, G)\)) iff \( f \) is injective and \( F = \{ f: A \rightarrow X : k \circ f \in G \} \).

(3) If \( (X, F, \text{id}) \) is a family of \( A \)-objects then the family of maps \( f: A \rightarrow \prod_{i \in I} X_i \) satisfying the condition that each component \( p_i \circ f \) belongs to \( F_i \) is the product structure \( \prod_{i \in I} F_i \), and the family \( \{ k_i \circ f : i \in I \text{ and } f \in F \} \), where \( k_i: X_i \rightarrow \prod_{i \in I} X_i \) are the canonical injections, is the coproduct structure \( \coprod_{i \in I} X_i \).

(4) \( q: (X, F) \rightarrow (Y, G) \) is a quotient morphism (and \((Y, G) \) a quotient space of \((X, F)\)) iff \( G = \{ q \circ f : i \in F \} \).

(5) \((X, F)\) is discrete iff \( F = \text{Set}(A, X) \).

(6) \((X, F)\) is indiscrete iff \( F = \{ \text{all constant maps} \} \).

2.4. Theorem. \( A \) is a strongly topological category, i.e.

(a) \( A \) is cartesian closed.

(b) Final epi-sinks in \( A \) are hereditary.

Proof. (a): It is to be shown ([2] Theorem 2 (1)\( \Leftrightarrow \) (6)) that, for each final epi-sink \( (h_i: (X_i, F_i) \rightarrow (X, F))_{i \in I} \) and \( A \)-object \((Y, G)\), the epi-sink \( (h_i \times 1_Y: (X_i \times Y, F_i \times G) \rightarrow (X \times Y, F \times G))_{i \in I} \) is final.

Let \( m \) be an element of \( F \times G \). Since the component \( m_X \) belongs to \( F \) (2.3. (3)), by 2.3. (1) there exist \( i \in I \) and \( f_1 \in F_1 \) with \( m_X = h_i \circ f_1 \). Since the component \( m_Y \) belongs to \( G \) (2.3. (3)) then the map \( \langle f_1, m_Y \rangle: A \rightarrow X_i \times Y \) belongs to \( F_i \times G \) and \( h_i \circ \langle f_1, m_Y \rangle = g \).

(b): Let \( (h_i: (X_i, F_i) \rightarrow (X, F))_{i \in I} \) be a final epi-sink in \( A \) and \( X' \) a subset of \( X \) with \( k: X' \rightarrow X \) the inclusion map. Further let \( F' \) be the subspace structure in \( X' \), and, for each \( i \in I \), let \( F'_i \) be the subspace structure in \( h_i^{-1}(X') \cap X_i \). It follows from 2.3. (2) and 2.3. (1) that the epi-sink \( (h_i / h_i^{-1}(X'): (h_i^{-1}(X'), F'_i) \rightarrow (X', F'))_{i \in I} \) is final.
2.5. Remark. If \( A \) is a two-point set then the category \( A \) coincides with the category \( \text{Rere} \) of all reflexive relations described in [3].

The subcategory \( \text{SRere} \) of all symmetric reflexive relations properly contains the trivial subcategory \( \text{Discrete}(\text{Rere}) \) of all discrete \( \text{Rere} \)-objects. Moreover the functors \( T_p : \text{Rere} \to \text{SRere} \) which respectively send each reflexive relation \( P \) into the smallest symmetric relation containing \( P \) and into the largest symmetric relation contained in \( P \), are left and right adjoint of the inclusion functor \( J : \text{SRere} \to \text{Rere} \).

Thus \( \text{SRere} \) is a proper non-trivial subcategory of \( \text{Rere} \) which is both bireflective and coreflective.

For each surjective map \( q : A \to B \) let \( A_q \) denote the full subcategory of \( A \) whose objects are all \( (X,F) \) satisfying the following condition: for every \( f \in F \) there exists \( g : B \to X \) such that \( g \circ q = f \).

2.6. Theorem. For each surjective map \( q : A \to B \) the category \( A_q \) is an epireflective and coreflective subcategory of \( A \). Moreover, if \( |A| > 2 \) and \( q \) is not injective, then \( A_q \) is a proper and non-trivial subcategory of \( A \).

Proof. Let \( (X,F) \) be an \( A \)-object. If \( cF = \{f \in F : \) there exists \( g : B \to X \) with \( g \circ q = f \} \), then the \( A \)-morphism \( c : (X,cF) \to (X,F) \), which is the identity map on the underlying set \( X \), is the coreflection of \( (X,F) \) in \( A_q \). Let \( rX \) be the quotient of \( X \) given by the equivalence relation generated by \( x \sim y \) iff there exist \( a,b \in A \) and \( f \in F \) such that \( fa = x, fb = y \) and \( qa = qb \) (i.e. \( rX \) is the largest quotient set of \( X \) such that, for each \( f \in F \) there exists \( g : B \to rX \) with \( g \circ q = r \circ f \), where
2.7. Remarks. (1) For each surjective map \( q: A \rightarrow B \) the functor \( Q: B \rightarrow A \) defined by \( Q(X,f,P) = (X,f,P \circ q) \) is a full embedding and \( QB = \mathbb{A}_q \). Then the theorem above says that the functors \( P_1, P_2: A \rightarrow B \), defined by \( P_1(Y,F) = (Y,\{h: B \rightarrow Y : h \circ q \circ g\}) \) and \( P_2(Y,F) = (rY,\{h: B \rightarrow Y : \exists f \in F \text{ with } h \circ q = r \circ f\}) \), are respectively left and right adjoint of the functor \( Q \).

In particular if \( B \) is a two-point set (that is \( B = \mathbb{Rere} \)) then each surjective map \( q: A \rightarrow B \) induces a full embedding \( Q: \mathbb{Rere} \rightarrow A \) as well as an epireflective and coreflective subcategory.

(2) If \( |A| \geq 3 \) and \( q: A \rightarrow B \) is not injective then \( \mathbb{A}_q \) is not bireflective in \( A \). It follows from (1) and Remark 2.5 that every \( A \), with \( |A| \geq 2 \) has a proper and non trivial, both bireflective and coreflective subcategory (namely \( \mathbb{SRere} \)).

(3) \( T_0 \)-objects in \( \mathbb{Rere} \) (i.e. objects \( (X,F) \) such that each \( \mathbb{Rere} \)-morphism of a two-point indiscrete object into \( (X,F) \) is constant) are the reflexive antisymmetric relations. Furthermore it follows from [2] Prop. 5 that a reflexive relation belongs to the bireflective hull \( IT_0 \mathbb{Rere} \) of the subcategory \( T_0 \mathbb{Rere} \) of all \( T_0 \)-objects iff the largest symmetric relation which is contained in it is an equivalence relation. Since it is very easy to find relations without the property above then \( \mathbb{Rere} \) is not universal in the sense of Marny ([8]).

(4) For each set \( A \) with \( |A| \geq 3 \) the topological category \( A \) is not universal. In fact if \( A \) would be a universal topologi-
A-object \((X,F)\) would be indiscrete iff \(R^X = \{(x,y) \in X \times X : \text{the subspace } \{x,y\} \text{ is indiscrete} \} = X \times X.\) But if we choose \((X,F)\) with \(|X| \geq 3\) and \(F = \{f : A 	o X : |\text{Im} f| \leq 2\},\) then \((X,F)\) is not indiscrete, and \(R^X = X \times X.\)

(5) The previous characterization of the class \(\text{Ind}\) of all indiscrete objects still holds for the category \(\text{ReRe}\) even if \(\text{ReRe}\) is not universal.

References


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(Oblatum 23.3. 1983)