MONOMORPHISMS AND EPIMORPHISMS OF INVERSE SYSTEMS
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Abstract: Monomorphisms and epimorphisms in a category Pro-\(C\) are studied. Characterizations of such morphisms are obtained in case \(C = \text{SET}\) or \(C\) is a topological category over \(\text{SET}\).


Key-words: inverse system, Pro-category, topological functor, pro-reflective subcategory.

0. INTRODUCTION. Given a category \(C\), Pro-\(C\) denotes the category of inverse systems in \(C\) and their morphisms, following Grothendieck’s definition [6]. The notion of inverse systems and the pro-categories have been widely used in Algebraic Topology and, after the work of Mardešić and Segal [10,11], they are a fundamental tool in the study of Shape Theory, in all its aspects. Nevertheless, there exist, up to author’s knowledge, no characterizations of monomorphisms and epimorphisms in Pro-\(C\) yet.

In this note we give some necessary and sufficient conditions in order to recognize special morphisms in that category. We shall be mainly concerned with those (Pro-\(C\))-morphisms having as domain or codomain a rudimentary system, i.e. a system formed by a single object of \(C\). Such morphisms are interesting since they play a central role in Shape Theory and in recent investigations in Categorical Topology, concerning the connections between (epi-) reflective

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and (epi-) pro-reflective subcategories [3,4,5,12,13].

Most of the results of the paper are contained in section 2 where we characterize monomorphisms and epimorphisms of Pro-SET having rudimentary domain or codomain; then we extend those results to any topological category over SET [7,8]. This is possible since the following holds: if U:C→SET is a topological functor, so is its extension Pro-U:Pro-C→Pro-SET which, therefore, preserves and reflects monomorphisms and epimorphisms.

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1. NOTATIONS AND PRELIMINARY RESULTS. The main reference for this note is Ch.I of [11]. The categorical terminology comes from [9].

DEFINITIONS. Let C be any category.

1.1. An inverse system \( K = (K_i, p_{ij}, I) \) in C is a collection \( \{K_i\}_{i \in I} \) of C-objects indexed over a directed set \((I, \leq)\), endowed with bonding morphisms \( p_{ij}: K_j \rightarrow K_i \), whenever \( i \leq j \), in such a way that \( p_{ii} = \text{identity} \) and \( p_{ij} \circ p_{jk} = p_{ik} \) for \( i \leq j \leq k \).

1.2. A morphism \( p: X \rightarrow K \) from a C-object \( X \) (== rudimentary system) to a system \( K \), is given by a family \( \{ p_i: X \rightarrow K_i \}_{i \in I} \) of C-morphisms, such that \( p_{ij} \circ p_i = p_j \), for all \( i \leq j \); then \( p \) is a natural source in C ([9],p.133).

1.3. A morphism \( q: H \equiv Y \) from a system \( H = (H, q_a, A) \) to a C-object \( Y \) (= rudimentary system), is an equivalence class of C-morphisms from some \( H \) to \( Y \). \( q_a: H \equiv Y \) and \( q_b: H \equiv Y \) are two representatives of \( q \) iff there is a \( c \in A \), \( c \geq a,b \), such that \( q_a \circ q_{ac} = q_b \circ q_{bc} \).

Let us call a morphism with rudimentary codomain \( q: H \equiv Y \) full iff it admits a representative \( q^*: H \equiv Y \), for all \( a \in A \).

1.4. A morphism of systems \( f: H \rightarrow K \) is a family \( \{ f_i: H \rightarrow K_i \}_{i \in I} \) of morphisms of type 1.3., such that \( f_{ij} = p_{ij} \circ f_i \), whenever \( i \leq j \). The composition is defined in the obvious way.
Inverse systems in $C$ and their morphisms form the category $\text{Pro-C}$.

1.5. Given two $(\text{Pro-C})$-morphisms $f: H \to K$ and $g: K \to R$, we can define their composition $g \cdot f$ to be given by the natural source in $\text{Pro-C}$ \( g \cdot f : H \to K \to R \mid c \in C \). Also, we can think of $g \cdot f$ as the natural source in $\text{Pro-C}$ \( \{ g \cdot f \psi(c) : H \to K \to R \mid c \in C \} \), where $g_c$ is a representative of $g$ and where $\psi : C \to I$ is a suitable function.

1.6. A pre-order $(I, \leq)$ is cofinite provided for all $i \in I$ the set \( \{ i \in I \mid i \leq j \} \) of its predecessors, is finite.

An inverse system with cofinite index set will be called a cofinite system.

**PROPOSITION 1.7.** Let $q': H' \to Y$. There exist an isomorphism $h : H \to H'$ in $\text{Pro-C}$ and a full morphism $q : H \to Y$ such that $q = q' \cdot h$ and $H$ is cofinite.

Proof. Let $H' = (H', q', A')$ and let $\tilde{A}$ denote the subset of $A'$ of all those indexes $a \in A'$ for which there exists a representative $q_i : H' \to Y$ of $q'$. Since $\tilde{A}$ is a directed cofinal subset of $A'$, if we let $\bar{H}$ denote the subsystem of $H'$ indexed over $\tilde{A}$, then the restriction morphism ([11], p.8) $h' : \bar{H} \to H'$ is an isomorphism in $\text{Pro-C}$. To conclude apply Theor.2, p.10 of [11] to obtain an isomorphism $h : \bar{H} \to \bar{H}$, with $\bar{H}$ cofinite, and put $q = q' \cdot h$, where $h = h' \cdot h$.

1.8. By the preceding result, every time we are given a $(\text{Pro-C})$-morphism with rudimentary codomain $q : H \to Y$, we may suppose, without loss of generality, that $H$ is cofinite and $q$ is full.

As a consequence, for every such $q$ we can choose a natural sink $q^* = (q_a : H \to Y \mid a \in A)$ of representatives of $q$. If $a \in A$, take a unique representative $q_{a_i}$ of $q$ for each predecessor $a_i$ of $a$ and let $q_a$ be the common value of the compositions $q_{a_i} \cdot q_{a \setminus a_i}$, for all $i$.

We say that $q^*$ is a sink representing the $(\text{Pro-C})$-morphism $q$. Obviously $q$ does not determine $q^*$ uniquely. If $q^* = (q'_a)$ is another sink representing $q$, then for all $a \in A$ there is $b \in A$ such that $q'_a = q'_b$. We express this fact by saying that the sinks $q^*$ and $q^*$ are cofinally equal or, simply, cofinal.

One easily verifies that if $q^* , q^*$ are sinks representing $(\text{Pro-C})$-morphisms $q , \tilde{q} : H \to Y$, respectively, then $q^*$ and $q^*$ cofinal implies $q = \tilde{q}$.

1.9. Recall from [7,8] that a functor $U: C \to D$ is topological if it admits
all initial (and final) liftings. In particular, every topological functor is
cotopological and preserves and reflects monomorphisms and epimorphisms \([7,8]\).

A concrete category \(C = (C, U: C \rightarrow \text{SET})\) is a topological category over \(\text{SET}\) when
the forgetful functor \(U\) is topological.

In the sequel \(H, K\) and \(R\) will always denote inverse systems \(H = (H, q, A), K = (K, p, I)\) and \(R = (R, r, C)\), unless otherwise specified.

2. MONOMORPHISMS AND EPIMORPHISMS WITH RUDIMENTARY DOMAIN OR CODOMAIN.

**DEFINITION 2.1.** Let \((A, \leq)\) be a directed set. A sink \(\{q_a : H \rightarrow Y \mid a \in A\}\) is said to be an epico
cofinal sink iff the following holds:
given \(f, g: Y \rightarrow Z\) with the property that for all \(a \in A\) there is an index \(b \geq a\) such
that \(f \circ q_b = g \circ q_b\), then \(f = g\).

Every epico
cofinal sink is an episink (\([9], \text{p.127}\)).

If \(q: H \rightarrow Y\) has a representing epico
cofinal sink, then every sink representing \(q\) is epico
cofinal.

**THEOREM 2.2.** \(q: H \rightarrow Y\) is an epimorphism in \(\text{Pro-C}\) iff \(q\) has a representing
epico
cofinal sink.

**Proof.** Let \(f, g: Y \rightarrow K\) be such that \(f \circ q = g \circ q\). This means that \(f \circ q = g \circ q\)
for all \(i \in I\). Let \(q^* = \{q_a : a \in A\}\) be an epico
cofinal sink representing \(q\); the sinks \(f \circ q^*\) and \(g \circ q^*\) are cofinal since they represent the same \((\text{Pro-C})\)-
morphism, hence
for all \(a \in A\) there is an index \(b \geq a\) such that \(f \circ q_b = g \circ q_b\), so that \(f \circ q^* = g \circ q^*,\)
by the hypothesis on \(q^*\), and this is true for all \(i \in I\). It follows \(f = g\) and \(q^*\) is an
epimorphism.

Let now \(q\) be an epimorphism in \(\text{Pro-C}\) and let \(q^*\) be a sink which represents \(q\).
Let \(f, g: Y \rightarrow Z\) be \(C\)-morphisms such that for all \(a \in A\) there is a \(b \geq a\) with \(f \circ q_b = g \circ q_b\).
Since \(f \circ q_b\) and \(g \circ q_b\) represent, respectively, \(f \circ q\) and \(g \circ q\), it follows
that \(f \circ q = g \circ q\), hence \(f = g\), since \(q\) is an epimorphism. Then \(q^*\) is epico
cofinal.

**PROPOSITION 2.3.** In \(\text{SET}\) one has:

i) \(\{p_i : X \rightarrow K_1 \mid i \in I\}\) is a monosource iff it separates points of \(X\).

ii) \(\{q_a : a \in A\}\) is an episink iff it covers points of \(Y\), i.e. for all
As we have already seen, an epic-final sink is a particular episink, hence, in SET it covers points of the codomain. The next theorem shows that a natural sink \( q \) in SET is epic-final iff it covers points in a special way.

**Theorem 2.4.** A natural sink \( q = (q_a) \) in SET is epic-final iff the following property holds:

(a) for every \( y \in Y \) there exists \( a \in A \) such that \( q^{-1}_a(y) \neq \emptyset \), for all \( b \geq a \).

Proof. Let \( q \) be a sink which satisfies condition (a); let \( f, g : Y \to Z \) be maps such that for all \( a \in A \) there is \( b \geq a \) with \( f \circ q_b = g \circ q_b \). It is easy to verify that in this situation one has \( f \circ q_c = g \circ q_c \) for all \( c \geq b \), too. Let us prove that \( f = g \). If \( y \) is any point in \( Y \), then there exists \( a \in A \) such that \( q^{-1}_b(y) \neq \emptyset \) for all \( b \geq a \). It is possible to choose \( b \) in order to have \( f \circ q_b = g \circ q_b \), at the same time. Then \( f(y) = (f \circ q_a)(h) = (g \circ q_a)(h) = g(y) \), for some \( h \in q^{-1}_b(y) \).

Suppose now that \( q \) is an epic-final sink and that (a) does not hold. Then there exists \( y_0 \in Y \) such that for every \( a \in A \) there is \( b(a) \geq a \) with \( q^{-1}_b(y_0) = \emptyset \).

Define two maps \( f, g : Y \to Y \) as follows. \( f = 1 \), and, if \( Y' = \bigcup_{a \in A} \text{Im } q_a \), let \( g|_{Y'} = \text{identity}, \quad g|_{Y - Y'} = \text{constant map of value } \overline{y} \in Y'. \) Then \( f \) and \( g \) are two maps which agree, at \( 1 : a \in A \), on \( y_0 \) and with the property that for every \( a \in A \) there is \( b(a) \geq a \) such that \( f \circ q_{b(a)} = q \circ q_{b(a)} \). But this last equality, by epicfinality of \( q \), implies \( f = g \), which is a contradiction.

**Corollary 2.5.** \( g : H \to Y \) is an epimorphism in Pro-SET iff \( g \) admits a representing sink \( q \) which satisfies condition (a) above.

Also (Pro-SET)-epimorphisms with rudimentary domain have a nice characterization.

**Theorem 2.6.** \( p : X \to K \) is an epimorphism in Pro-SET iff for all \( i \in I \) such that \( p_i : Y \to K \) is not onto, there exists an index \( j \geq i \) with \( \text{Im } p_{ij} \subset \text{Im } p_i \).

Proof. If \( p_i \) is onto for all \( i \in I \), then \( p \) is an epimorphism in Pro-SET.

Suppose that \( p \) is an epimorphism in Pro-SET and let \( p_i : X \to K \) be not surjective. Let us consider maps \( f, g : K \to K \), given by \( f = 1 - k, \quad g|_{\text{Im } p_i} = \text{identity}, \quad g|_{K - \text{Im } p_i} \). Then \( f = g \), and thus \( p_i \) is onto for all \( i \in I \).
Since $f_i$ and $g_i$ represent $(\text{Pro-SET})$-morphisms $f, g: K \to K_i$ and since $f_i p_i = g_i p_i$, it follows that $f p = g p$, so that $f = g$, $p$ being an epimorphism. This equality means that there is a $j \geq i$ such that $f_i p_{ij} = g_i p_{ij}$, which implies $\text{Im } p_{ij} \subseteq \text{Im } p_i$, since, by definition $\text{Im } g_i = \text{Im } p_i$.

Conversely, let $p: X \rightarrow K$ be a $(\text{Pro-SET})$-morphism with the property that for all $i$ in $I$ such that $p_i$ is not onto, there is a $j \geq i$ with $\text{Im } p_{ij} \subseteq \text{Im } p_i$. Let us prove that $p$ is an epimorphism in $\text{Pro-SET}$. Let $f, g: K \rightarrow H$ be such that $f p = g p$.

We may suppose, without any restriction, that $K$ and $H$ are indexed over the same directed set $I$ and that $f, g$ admit as representatives the level maps $(f_i, 1)$, $(g_i, 1)$, respectively ([11], Th.3.3). Then, from $f p = g p$ one obtains that $f_i p_i = g_i p_i$, for all $i \in I$. If $p_i$ is not onto, let $j \geq i$ be in the hypothesis: $p_{ij} p_j = p_i$; hence $f_i p_{ij} p_j = g_i p_{ij} p_j$. Now, let $x \in K_j$, then there exists an $x \in X$ such that $p_{ij}(x) = p_i(x)$; it follows $(f_i p_{ij})(x) = f_i(p_i(x)) = g_i(p_i(x)) = (g_i p_{ij})(x)$, hence $f_i p_{ij} = g_i p_{ij}$, that is $f = g$ in $\text{Pro-SET}$. Hence $p$ is an epimorphism.

**Proposition 2.7.** $p: X \rightarrow K$ is a monomorphism in $\text{Pro-SET}$ iff the source $p_i$ separates points of $X$.

Proof. By Proposition 2.3.11).

**Theorem 2.8.** $q: H \rightarrow Y$ is a monomorphism in $\text{Pro-SET}$ iff there exists a sink $q = (q_a)$ representing $q$ which satisfies the following property:

(8) for every $a \in A$ there is $b \geq a$ such that $q_b$ is injective.

Proof. Let us prove that condition (8) is sufficient. Let $f, g: K \rightarrow H$ be such that $q f = q g$. We may suppose, as in the proof of 2.6., that $H$ and $K$ are indexed over the same directed set $A$, $K = (K_a)_{a \in A}$, and that $f, g$ are represented by level maps $(f_a, 1)$, $(g_a, 1)$, respectively. In this case the sinks $\{ q_a f_a : K \rightarrow Y \mid a \in A \}$ and $\{ q_a g_a : K \rightarrow Y \mid a \in A \}$ must be cofinal, since they represent the same $(\text{Pro-SET})$-morphism with rudimentary codomain; hence for every $a \in A$ there is $c \geq a$ such that $q_a f_a = q_c g_{c}$. Let now $d \in A$, $d \geq a, b, c$ and consider the following diagram:

$$
\begin{array}{cccc}
q_{a} & f_{a} & K & \rightarrow & Y \\
\downarrow & & \| & & \downarrow \\
q_{c} & g_{c} & K & \rightarrow & Y
\end{array}
$$
which gives \( q \cdot f \cdot p_{bd} = q \cdot f \cdot p_{ad} = q \cdot f \cdot p_{cd} \) and \( q \cdot g \cdot p_{bd} = q \cdot g \cdot p_{ad} \).

By the assumption that \( q \cdot f \) is a monomorphism, it follows \( f \cdot p_{bd} = q \cdot g \cdot p_{bd} \) and also \( q \cdot f \cdot p_{bd} = q \cdot g \cdot p_{bd} \), since \( q \cdot g \) is a monomorphism. Finally, one has \( f \cdot p_{ad} = a \cdot p_{ad} \).

Consider the level maps \((f, 1)\), \((g, 1)\) represent the same \((\text{Pro-SET})\)-morphism, that is \( f = g \) and \( q \) is a monomorphism.

Conversely, let \( q: H \to Y \) be a monomorphism in \( \text{Pro-SET} \) and let \( \mathbf{q} = (q_A) \) be a sink representing \( q \). Suppose that \( \mathbf{q} \) does not satisfy condition \((\beta)\), then there exists an index \( a_0 \) in \( A \) such that for all \( b \geq a_0 \), \( q_b \) is not injective. Hence, for all \( b \geq a_0 \), there are \( h'_b \neq h''_b \) in \( H_b \) such that \( q_b(h'_b) = q_b(h''_b) \). Define maps \( f, g: H \to H \), for all \( a \in A \), as follows: \( f = 1 \), \( g = 1 \), for all \( a \in A \), and if \( A = \{ b \in A | b \geq a \} \), let \( q_a = 1 \), for all \( a \in A \), while, for \( b \in A \), let \( q_b: H_b \to H_b \) be the map which permutes \( h'_b \) and \( h''_b \) and leaves all other points fixed.

Moreover, for every \( a \leq b \) in \( A \) let \( q_{ab}: H_b \to H_a \) be a map which makes the following diagram commutative

\[
\begin{array}{ccc}
H & \xrightarrow{q_b} & H \\
\downarrow {q_{ab}} & & \downarrow {q_{ab}} \\
H & \xrightarrow{q_a} & H \\
\end{array}
\]

Note that \( q_{ab} \) will act the same as \( q_{ab} \) up to a rearrangement of its values on \( h'_b, h''_b, q^{-1}(h'_b), q^{-1}(h''_b) \), when needed.

From the above it follows that \((H, q_{ab}, A)\) is an inverse system in \( \text{SET} \), denoted by \( \prod H \), while \((f, 1), (g, 1)\) are level maps which represent \((\text{Pro-SET})\)-morphisms.
Let now \((C, U: C \rightarrow \text{SET})\) be a concrete category and let \(\text{Pro-}U: \text{Pro-C} \rightarrow \text{Pro-SET}\), \(K \mapsto \text{UK} = (\text{UK}, \text{Up}, I)\), be the extension of \(U\) to the pro-categories.

**Theorem 2.9.** If \((C, U: C \rightarrow \text{SET})\) is a topological category over \(\text{SET}\), then \(\text{Pro-}U\) is a topological functor.

**Proof.** It suffices to prove that every \((\text{Pro-}U)\)-sink \(\{f^a: \text{UK}^a \rightarrow S | a \in A\}\), \(S = (S, s, A) \in \text{Pro-SET}\), has a unique \((\text{Pro-}U)\)-final lifting; that is there exists a unique, up to isomorphisms, \(H \in \text{Pro-C}\) and \(a(\text{Pro-}U)\)-final sink \(\{g^a: K^a \rightarrow H | a \in A\}\), such that \(S = UH\) and \(g^a = Uf^a\), for all \(a \in A\).

We only sketch the construction of \(H\), leaving all other easy details to the reader. For every \(a \in A\), let \(\{f^a_{\phi(a)}: \text{UK}^a \rightarrow S | a \in A\}\) be the sink where \(f^a_{\phi(a)}\) is a representative of \(f^a: \text{UK}^a \rightarrow S\), and let \(\{g^a_{\phi(a)}: K^a \rightarrow H | a \in A\}\) be its \(U\)-final lifting, which there exists by hypothesis. Then, by the properties of the final lifting, it follows that, for all \(a \leq b\), \(s_{ab}: S_b \rightarrow S_a\) comes from a certain \(C\)-morphism \(q_{ab}: H_b \rightarrow H_a\), so that \(H = (H, q_{ab}, A)\) is an inverse system in \(C\).

By virtue of the above theorem, since topological functors (1.9) preserve monomorphisms and epimorphisms, all the results in this section, characterizing monomorphisms and epimorphisms in \(\text{Pro-SET}\), can be extended to \(\text{Pro-C}\) as well, where \(C\) is any topological category over \(\text{SET}\).

**Remarks.**

2.10. Let \(K\) be an inverse system in \(\text{HComp}\), the category of compact Hausdorff
spaces, and let \( \varphi: X \rightarrow K \) be the inverse limit morphism [1]. If \( \varphi_i(K) \) is an open
set in \( K_i \), for all \( i \), then \( \varphi \) is an epimorphism in Pro-HComp. This follows from
[2], Th.3.7, p.217, and Th. 2.6. above.

2.11. Let HComp be the category of locally compact Hausdorff spaces and
let TYCH be the category of completely regular \( T_\infty \) spaces. Every \( X \in TYCH \) admits
an HComp-expansion \([3,11] \) \( \varphi: X \rightarrow K \), where \( K \in \text{Pro-HComp} \) is formed by taking
all open neighbourhoods of \( X \) in \( \beta X \), its Stone-Čech compactification, directed by
reversed inclusion. \( \varphi \) is an epimorphism in Pro-TYCH, in fact each \( \varphi_i \) is an
epimorphism in TYCH.

If \( S \) is any topological space and \( \eta: S \rightarrow X \) is its epireflection in TYCH, then
\( \varphi \circ \eta: S \rightarrow K \) is an HComp-expansion of \( S \) which is not an epimorphism in
Pro-TOP (TOP being the category of all topological spaces and continuous maps)
This may be seen using Th. 2.6.

3. GENERAL MONOMORPHISMS AND EPIMORPHISMS IN Pro-\( C \). Let now \( C \) be an
arbitrary category.

**Lemma 3.1.** Let \( e: H \rightarrow K, f, g: K \rightarrow R \) be given morphisms in Pro-\( C \). Then
\( f \circ e = g \circ e \) holds iff there is an index \( i \in I \) such that \( f_i \circ e_i = g_i \circ e_i \), where \( f_i, g_i \)
represent \( f, g \), respectively.

**Proof.** Follows from the definitions and from 1.5.

**Proposition 3.2.** \( e: H \rightarrow K \) is an epimorphism in Pro-\( C \) iff there is an index
\( i \in I \) such that \( e_i: H_i \rightarrow K_i \) is an epimorphism, i.e. has a representing epicofinal
sink.

**Proof.** Let \( e_i \) be an epimorphism. If \( f_i, g_i: K_i \rightarrow R \) are such that \( f_i \circ e_i = g_i \circ e_i \),
then \( f \circ e = g \circ e \) for all c.e.c, which means \( f \circ i \circ e = g \circ i \circ e \), by the Lemma, hence
\( f \circ i = g \circ i \), for all c.e.c. It follows \( f = g \), and \( e \) is an epimorphism.

Conversely, let \( e \) be an epimorphism and suppose that no \( e_i, i \in I \), is an epimorphism.
Then, for all \( i \in I \), there are \( f \neq g: K_i \rightarrow R \) such that \( f \circ e_i = g \circ e_i \). This last
implies \( f \circ e = g \circ e \), for all c.e.c, then \( f \circ e = g \circ e \), for all c.e.c, by the Lemma.
By assumption one obtains \( f \circ c = g \circ c \), for all c.e.c, that is \( f = g \), which is a
contradiction.

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PROPOSITION 3.3. If $m: K \to R$ is such that $m_c: K \to R$ is a monomorphism, for some $c \in C$, then also $m$ is a monomorphism in $\text{Pro-} C$.

Proof. Suppose $m_c$ is a monomorphism in $\text{Pro-} C$ and let $f, g: H \to K$ be such that $m_c f = m_c g$. This means $m_c^{-1} f = m_c^{-1} g$, for all $d \in C$, then $f = g$.

This proposition may be inverted in some cases, such as the following.

PROPOSITION 3.4. Let $m: K \to R$ be a monomorphism in $\text{Pro-} C$. If there is an index $c \in C$ with the property that $r_d: R \to R$ is a monomorphism in $C$ for every $d \geq c$, then $m: K \to R$ is a monomorphism in $\text{Pro-} C$.

Proof. Let $f, g: H \to K$ be such that $m_c f = m_c g$. Then, for all $e \in C$, one has the following commutative diagram, where $d \geq c$,

Since $r_d$ is a monomorphism, $m_e f = m_e g$. Finally $f = g$ by the assumption and by 1.5.

Note that, in case $C = \text{SET}$ or $C$ is a topological category over $\text{SET}$, then one can apply the results of section 2 in order to obtain information about monomorphisms and epimorphisms in $\text{Pro-} C$ of the general form $H \to K$.

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