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MONOMORPHISMS AND EPIMORPHISMS OF INVERSE SYSTEMS

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Abstract: Monomorphisms and epimorphisms in a category Pro-$\mathcal{C}$ are studied. Characterizations of such morphisms are obtained in case $\mathcal{C}$ = SET or $\mathcal{C}$ is a topological category over SET.

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0. INTRODUCTION. Given a category $\mathcal{C}$, Pro-$\mathcal{C}$ denotes the category of inverse systems in $\mathcal{C}$ and their morphisms, following Grothendieck's definition [6]. The notion of inverse systems and the pro-categories have been widely used in Algebraic Topology and, after the work of Mardešić and Segal [10,11], they are a fundamental tool in the study of Shape Theory, in all its aspects. Nevertheless, there exist, up to author's knowledge, no characterizations of monomorphisms and epimorphisms in Pro-$\mathcal{C}$ yet.
In this note we give some necessary and sufficient conditions in order to recognize special morphisms in that category. We shall be mainly concerned with those (Pro-$\mathcal{C}$)-morphisms having as domain or codomain a rudimentary system, i.e. a system formed by a single object of $\mathcal{C}$. Such morphisms are interesting since they play a central role in Shape Theory and in recent investigations in Categorical Topology, concerning the connections between (epi-) reflective

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and (epi-) pro-reflective subcategories [3,4,5,12,13].

Most of the results of the paper are contained in section 2 where we characterize monomorphisms and epimorphisms of Pro-SET having rudimentary domain or codomain; then we extend those results to any topological category over SET [7,8]. This is possible since the following holds: if $U: C \to SET$ is a topological functor, so is its extension Pro-U: Pro-C $\to$ Pro-SET which, therefore, preserves and reflects monomorphisms and epimorphisms.

Thanks are due to the referee for having suggested the last result and for many valuable advices about the general arrangement of the paper.

1. NOTATIONS AND PRELIMINARY RESULTS. The main reference for this note is Ch.I of [11]. The categorical terminology comes from [9].

DEFINITIONS. Let C be any category.

1.1. An inverse system $\underline{K} = (K_i, p_{ij}, I)$ in C is a collection $\{K_i\}_{i\in I}$ of C-objects indexed over a directed set $(I, \leq)$, endowed with bonding morphisms p_{ij}: $K_j \to K_i$, whenever $i \leq j$, in such a way that p_{ii} = identity and $p_{ij} \cdot p_{jk} = p_{ik}$, for $i \leq j \leq k$.

1.2. A morphism $\underline{p}$: $X \to \underline{K}$ from a C-object X (= rudimentary system) to a system $\underline{K}$, is given by a family $\{ p_i : X \to K_i \mid i\in I \}$ of C-morphisms, such that $p_{ij} \cdot p_j = p_i$, for all $i \leq j$; then $\underline{p}$ is a natural source in C ([9],p.133).

1.3. A morphism $\underline{q}$: $\underline{H} \to Y$ from a system $\underline{H} = (H_a, q_{ab}, A)$ to a C-object Y (= rudimentary system), is an equivalence class of C-morphisms from some H_a to Y. $q_a : H_a \to Y$ and $q_b : H_b \to Y$ are two representatives of $\underline{q}$ iff there is a $c\in A$, $c \geq a,b$, such that $q_a \cdot q_{ac} = q_b \cdot q_{bc}$.

Let us call a morphism with rudimentary codomain $\underline{q}$: $\underline{H} \to Y$ full iff it admits a representative $q_a : H_a \to Y$, for all $a\in A$.

1.4. A morphism of systems $\underline{f}$: $\underline{H} \to \underline{K}$ is a family $\{ \underline{f}_i : \underline{H} \to K_i \mid i\in I \}$ of morphisms of type 1.3., such that $\underline{f}_i = p_{ij} \cdot \underline{f}_j$, whenever $i \leq j$. The composition is defined in the obvious way.

Inverse systems in $\mathcal{C}$ and their morphisms form the category Pro-$\mathcal{C}$.

1.5. Given two (Pro-$\mathcal{C}$)-morphisms $\underline{f}: \underline{H} \to \underline{K}$ and $\underline{g}: \underline{K} \to \underline{R} = (R_c, r_{cd}, C)$, we can define their composition $\underline{g}\cdot\underline{f}$ to be given by the natural source in Pro-$\mathcal{C}$ $\{\underline{g}_c \cdot \underline{f} : \underline{H} \to \underline{K} \to R_c \mid c \in C\}$. Also, we can think of $\underline{g}\cdot\underline{f}$ as the natural source in Pro-$\mathcal{C}$ $\{ g_c \cdot \underline{f}_{\psi(c)} : \underline{H} \to K_{\psi(c)} \to R_c \mid c \in C \}$, where g_c is a representative of $\underline{g}_c$ and where $\psi: C \to I$ is a suitable function.

1.6. A pre-order $(I, \leq)$ is cofinite provided for all $j \in I$ the set $\{i \in I \mid i \leq j\}$, of its predecessors, is finite.

An inverse system with cofinite index set will be called a cofinite system.

PROPOSITION 1.7. Let $\underline{q}': \underline{H}' \to Y$. There exist an isomorphism $\underline{h}: \underline{H} \to \underline{H}'$ in Pro-$\mathcal{C}$ and a full morphism $\underline{q}: \underline{H} \to Y$ such that $\underline{q} = \underline{q}'\cdot\underline{h}$ and $\underline{H}$ is cofinite.

Proof. Let $\underline{H}' = (H'_a, q'_{ab}, A')$ and let $\tilde{A}$ denote the subset of A' of all those indexes $a \in A'$ for which there exists a representative $q'_a: H'_a \to Y$ of $\underline{q}'$. Since $\tilde{A}$ is a directed cofinal subset of A', if we let $\underline{\tilde{H}}$ denote the subsystem of $\underline{H}'$ indexed over $\tilde{A}$, then the restriction morphism ([11], p.8) $\underline{h}': \underline{\tilde{H}} \to \underline{H}'$ is an isomorphism in Pro-$\mathcal{C}$. To conclude apply Theor.2, p.10 of [11] to obtain an isomorphism $\underline{\tilde{h}}: \underline{H} \to \underline{\tilde{H}}$, with $\underline{H}$ cofinite, and put $\underline{q} = \underline{q}'\cdot\underline{h}$, where $\underline{h} = \underline{h}'\cdot\underline{\tilde{h}}$.

1.8. By the preceding result, every time we are given a (Pro-$\mathcal{C}$)-morphism with rudimentary codomain $\underline{q}: \underline{H} \to Y$, we may suppose, without loss of generality, that $\underline{H}$ is cofinite and $\underline{q}$ is full.

As a consequence, for every such $\underline{q}$ we can choose a natural sink $q^* = \{q_a : H_a \to Y \mid a \in A\}$ of representatives of $\underline{q}$. If $a \in A$, take a unique representative q_{a_i} of $\underline{q}$ for each predecessor a_i of a and let q_a be the common value of the compositions $q_{a_i} \cdot q_{a_i a}$, for all i.

We say that q^* is a sink representing the (Pro-$\mathcal{C}$)-morphism $\underline{q}$. Obviously $\underline{q}$ does not determine q^* uniquely. If $\tilde{q}^* = (\tilde{q}_a)_A$ is another sink representing $\underline{q}$, then for all $a \in A$ there is $b \geq a$ such that $q_b = \tilde{q}_b$. We express this fact by saying that the sinks q^* and $\tilde{q}^*$ are cofinally equal or, simply, cofinal.

One easily verifies that if q^*, $\tilde{q}^*$ are sinks representing (Pro-$\mathcal{C}$)-morphisms $\underline{q}$, $\underline{\tilde{q}}: \underline{H} \to Y$, respectively, then q^* and $\tilde{q}^*$ cofinal implies $\underline{q} = \underline{\tilde{q}}$.

1.9. Recall from [7,8] that a functor $U: \mathcal{C} \to \mathcal{D}$ is topological if it admits all initial (and final) liftings. In particular, every topological functor is cotopological and preserves and reflects monomorphisms and epimorphisms [7,8]. A concrete category $\mathcal{C} = (C, U: C \to SET)$ is a <u>topological category</u> over SET when the forgetful functor U is topological.

In the sequel $\underline{H}$, $\underline{K}$ and $\underline{R}$ will always denote inverse systems $\underline{H} = (H_a, q_{ab}, A)$, $\underline{K} = (K_i, p_{ij}, I)$ and $\underline{R} = (R_c, r_{cd}, C)$, unless otherwise specified.

2. MONOMORPHISMS AND EPIMORPHISMS WITH RUDIMENTARY DOMAIN OR CODOMAIN.

<u>DEFINITION 2.1</u>. Let $A = (A, \leq)$ be a directed set. A sink $\{q_a: H_a \to Y \mid a \in A\}$ is said to be an <u>epicofinal sink</u> iff the following holds:
given $f, g: Y \to Z$ with the property that for all $a \in A$ there is an index $b \geq a$ such that $f \cdot q_b = g \cdot q_b$, then $f = g$.
Every epicofinal sink is an episink ([9], p.127).
If $\underline{q}: \underline{H} \to Y$ has a representing epicofinal sink, then every sink representing $\underline{q}$ is epicofinal.

<u>THEOREM 2.2</u>. $\underline{q}: \underline{H} \to Y$ is an epimorphism in Pro-C iff $\underline{q}$ has a representing epicofinal sink.

Proof. Let $\underline{f}, \underline{g}: Y \to \underline{K}$ be such that $\underline{f} \cdot \underline{q} = \underline{g} \cdot \underline{q}$. This means that $f_i \cdot \underline{q} = g_i \cdot \underline{q}$ for all $i \in I$. Let $q^* = (q_a)_A$ be an epicofinal sink representing $\underline{q}$; the sinks $f_i \cdot q^*$ and $g_i \cdot q^*$ are cofinal since they represent the same (Pro-C)-morphism, hence for all $a \in A$ there is an index $b \geq a$ such that $f_i \cdot q_b = g_i \cdot q_b$, so that $f_i = g_i$, by the hypothesis on q^*, and this is true for all $i \in I$. It follows $\underline{f} = \underline{g}$ and $\underline{q}$ is an epimorphism.
Let now $\underline{q}$ be an epimorphism in Pro-C and let q^* be a sink which represents $\underline{q}$. Let $f, g: Y \to Z$ be C-morphisms such that for all $a \in A$ there is a $b \geq a$ with $f \cdot q_b = g \cdot q_b$. Since $f \cdot q_b$ and $g \cdot q_b$ represent, respectively, $f \cdot \underline{q}$ and $g \cdot \underline{q}$, it follows that $f \cdot \underline{q} = g \cdot \underline{q}$, hence $f = g$, since $\underline{q}$ is an epimorphism. Then q^* is epicofinal.

<u>PROPOSITION 2.3</u>. In SET one has:

i) $\{p_i: X \to K_i \mid i \in I\}$ is a monosource iff it separates points of X.

ii) $\{q_a: H_a \to Y \mid a \in A\}$ is an episink iff it covers points of Y, i.e. for all

$y \in Y$ there are $a \in A$ and $h \in H_a$ with $y = q_a(h)$.

As we have already seen, an epicofinal sink is a particular episink, hence, in SET it covers points of the codomain. The next theorem shows that a natural sink q^* in SET is epicofinal iff it covers points in a special way.

THEOREM 2.4. A natural sink $q^* = (q_a)_A$ in SET is epicofinal iff the following property holds:

(α) for every $y \in Y$ there exists $a \in A$ such that $q_b^{-1}(y) \neq \emptyset$, for all $b \geq a$.

Proof. Let q^* be a sink which satisfies condition (α); let $f, g: Y \to Z$ be maps such that for all $a \in A$ there is $b \geq a$ with $f \cdot q_b = g \cdot q_b$. It is easy to verify that in this situation one has $f \cdot q_c = g \cdot q_c$ for all $c \geq b$, too. Let us prove that $f = g$. If y is any point in Y, then there exists $a \in A$ such that $q_b^{-1}(y) \neq \emptyset$ for all $b \geq a$. It is possible to choose b in order to have $f \cdot q_b = g \cdot q_b$, at the same time. Then $f(y) = (f \cdot q_b)(h) = (g \cdot q_b)(h) = g(y)$, for some $h \in q_b^{-1}(y)$.
Suppose now that q^* is an epicofinal sink and that (α) does not hold. Then there exists $y_0 \in Y$ such that for every $a \in A$ there is $b(a) \geq a$ with $q_{b(a)}^{-1}(y_0) = \emptyset$. Define two maps $f, g: Y \to Y$ as follows. $f = 1_Y$ and, if $Y' = \bigcup_{a \in A} \mathrm{Im}\, q_{b(a)}$, let $g_{|Y'}$ = identity, $g_{|Y-Y'}$ = constant map of value $\bar{y} \in Y'$. Then f and g are two maps which differ, at least, in y_0 and with the property that for every $a \in A$ there is $b(a) \geq a$ such that $f \cdot q_{b(a)} = g \cdot q_{b(a)}$. But this last equality, by epicofinality of q^*, implies $f = g$, which is a contradiction.

COROLLARY 2.5. $\underline{q}: \underline{H} \to Y$ is an epimorphism in Pro-SET iff $\underline{q}$ admits a representing sink q^* which satisfies condition (α) above.

Also (Pro-SET)-epimorphisms with rudimentary domain have a nice characterization.

THEOREM 2.6. $\underline{p}: X \to \underline{K}$ is an epimorphism in Pro-SET iff for all $i \in I$ such that $p_i: X \to K_i$ is not onto, there exists an index $j \geq i$ with $\mathrm{Im}\, p_{ij} \subset \mathrm{Im}\, p_i$

Proof. If p_i is onto for all $i \in I$, then $\underline{p}$ is an epimorphism in Pro-SET. Suppose that $\underline{p}$ is an epimorphism in Pro-SET and let $p_i: X \to K_i$ be not surjective. Let us consider maps $f_i, g_i: K_i \to K_i$, given by $f_i = 1_{K_i}$, $g_{i|\mathrm{Im}\, p_i}$ = identity,

$g_i|_{K_i - \mathrm{Im}\, p_i}$ = constant map of value $\bar{x}\in \mathrm{Im}\, p_i$.

Since f_i and g_i represent (Pro-SET)-morphisms $\underline{f}$, $\underline{g}$: $\underline{K} \to K_i$ and since $f_i \cdot p_i = g_i \cdot p_i$, it follows that $\underline{f}\cdot\underline{p} = \underline{g}\cdot\underline{p}$, so that $\underline{f} = \underline{g}$, $\underline{p}$ being an epimorphism. This equality means that there is a $j \geq i$ such that $f_i \cdot p_{ij} = g_i \cdot p_{ij}$, which implies $\mathrm{Im}\, p_{ij} \subset \mathrm{Im}\, p_i$, since, by definition $\mathrm{Im}\, g_i = \mathrm{Im}\, p_i$.

Conversely, let $\underline{p}: X \to \underline{K}$ be a (Pro-SET)-morphism with the property that for all i in I such that p_i is not onto, there is a $j \geq i$ with $\mathrm{Im}\, p_{ij} \subset \mathrm{Im}\, p_i$. Let us prove that $\underline{p}$ is an epimorphism in Pro-SET. Let $\underline{f}$, $\underline{g}$: $\underline{K} \to \underline{H}$ be such that $\underline{f}\cdot\underline{p} = \underline{g}\cdot\underline{p}$. We may suppose, without any restriction, that $\underline{K}$ and $\underline{H}$ are indexed over the same directed set I and that $\underline{f}$, $\underline{g}$ admit as representatives the level maps $(f_i, 1_I)$, $(g_i, 1_I)$, respectively ([11], Th.3.3). Then, from $\underline{f}\cdot\underline{p} = \underline{g}\cdot\underline{p}$ one obtains that $f_i \cdot p_i = g_i \cdot p_i$, for all $i\in I$. If p_i is not onto, let $j \geq i$ be as in the hypothesis: $p_{ij}\cdot p_j = p_i$; hence $f_i \cdot p_{ij} \cdot p_j = g_i \cdot p_{ij} \cdot p_j$. Now, let $z\in K_j$, then there exists an $x\in X$ such that $p_{ij}(z) = p_i(x)$; it follows $(f_i \cdot p_{ij})(z) = f_i(p_i(x)) = g_i(p_i(x)) = (g_i \cdot p_{ij})(z)$, hence $f_i \cdot p_{ij} = g_i \cdot p_{ij}$, that is $\underline{f} = \underline{g}$ in Pro-SET. Hence $\underline{p}$ is an epimorphism.

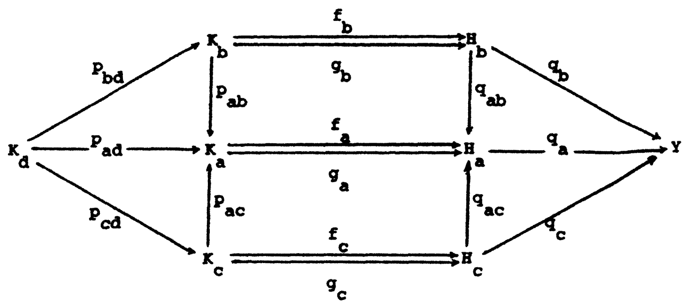
PROPOSITION 2.7. $\underline{p}$: $X \to \underline{K}$ is a monomorphism in Pro-SET iff the source $\underline{p} = (p_i)_I$ separates points of X.

Proof. By Proposition 2.3.i).

THEOREM 2.8. $\underline{q}$: $\underline{H} \to Y$ is a monomorphism in Pro-SET iff there exists a sink $q^* = (q_a)_A$ representing $\underline{q}$ which satisfies the following property:

(β) for every $a\in A$ there is $b \geq a$ such that q_b is injective.

Proof. Let us prove that condition (β) is sufficient. Let $\underline{f}$, $\underline{g}$: $\underline{K} \to \underline{H}$ be such that $\underline{q}\cdot\underline{f} = \underline{q}\cdot\underline{g}$. We may suppose, as in the proof of 2.6. , that $\underline{H}$ and $\underline{K}$ are indexed over the same directed set A, $\underline{K} = (K_a, p_{ab}, A)$, and that $\underline{f}$, $\underline{g}$ are represented by level maps $(f_a, 1_A)$, $(g_a, 1_A)$, respectively. In this case the sinks $\{ q_a \cdot f_a : K_a \to Y \mid a\in A \}$ and $\{ q_a \cdot g_a : K_a \to Y \mid a\in A \}$ must be cofinal, since they represent the same (Pro-SET)-morphism with rudimentary codomain; hence for every $a\in A$ there is $c \geq a$ such that $q_c \cdot f_c = q_c \cdot g_c$. Let now $d\in A$, $d \geq a,b,c$ and consider the following diagram

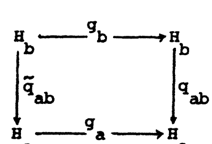


which gives $q_b \cdot f_b \cdot p_{bd} = q_a \cdot f_a \cdot p_{ad} = q_c \cdot f_c \cdot p_{cd}$ and $q_b \cdot g_b \cdot p_{bd} = q_a \cdot g_a \cdot p_{ad} = q_c \cdot g_c \cdot p_{cd}$.

By the assumption that $q_c \cdot f_c = q_c \cdot g_c$ it follows $q_b \cdot f_b \cdot p_{bd} = q_b \cdot g_b \cdot p_{bd}$ and also $f_b \cdot p_{bd} = g_b \cdot p_{bd}$, since q_b is a monomorphism. Finally, one has $f_a \cdot p_{ad} = f_a \cdot p_{ab} \cdot p_{bd} = q_{ab} \cdot f_b \cdot p_{bd} = q_{ab} \cdot g_b \cdot p_{bd} = g_a \cdot p_{ab} \cdot p_{bd} = g_a \cdot p_{ad}$. Hence we have shown that the level maps $(f_a, 1_A)$, $(g_a, 1_A)$ represent the same (Pro-SET)-morphism, that is $\underline{f} = \underline{g}$ and $\underline{q}$ is a monomorphism.

Conversely, let $\underline{q}: \underline{H} \to Y$ be a monomorphism in Pro-SET and let $q^* = (q_a)_A$ be a sink representing $\underline{q}$. Suppose that q^* does not satisfy condition (β), then there exists an index a_0 in A such that for all $b \geq a_0$, q_b is not injective. Hence, for all $b \geq a_0$, there are $h'_b \neq h''_b$ in H_b such that $q_b(h'_b) = q_b(h''_b)$. Define maps $f_a, g_a : H_a \to H_a$, for all $a \in A$, as follows: $f_a = 1_{H_a}$, for all $a \in A$, and if $A_0 = \{b \in A \mid b \geq a_0\}$, let $g_a = 1_{H_a}$, for all $a \in A - A_0$, while, for $b \in A_0$, let $g_b : H_b \to H_b$ be the map which permutes h'_b and h''_b and leaves all other points fixed.

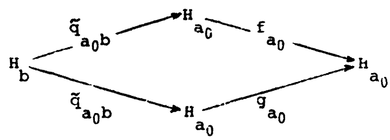
Moreover, for every $a \leq b$ in A let $\tilde{q}_{ab} : H_b \to H_a$ be a map which makes the following diagram commutative



Note that $\tilde{q}_{ab}$ will act the same as q_{ab} up to a rearrangement of its values on h'_b, h''_b, $q^{-1}_{ab}(h'_a)$, $q^{-1}_{ab}(h''_a)$, when needed.

From the above it follows that $(H_a, \tilde{q}_{ab}, A)$ is an inverse system in SET, denoted by $\underline{\tilde{H}}$, while $(f_a, 1_A)$, $(g_a, 1_A)$ are level maps which represent (Pro-SET)-morphisms

$\underline{f}, \underline{g}: \underline{\tilde{H}} \to \underline{H}$. One has $\underline{q}\cdot\underline{f} = \underline{q}\cdot\underline{g}$, in fact $q_a \cdot f_a = q_a \cdot g_a$, for all $a\varepsilon A$, hence $\underline{f} = \underline{g}$, since $\underline{q}$ is a monomorphism. But this last equality means that there exists $b \geq a_0$ such that the diagram



commutes, that is $g_{a_0} \cdot \tilde{q}_{a_0 b} = \tilde{q}_{a_0 b}$, which is impossible because of the definition of the maps involved. Hence condition (β) must hold when $\underline{q}$ is a monomorphism in Pro-SET.

Let now (C, $U:C \to$ SET) be a concrete category and let Pro-U: Pro-C $\to$ Pro-SET, $\underline{K} \mapsto \underline{UK} = (UK_i, Up_{ij}, I)$, be the extension of U to the pro-categories.

THEOREM 2.9. If (C, $U:C \to$ SET) is a topological category over SET, then Pro-U is a topological functor.

Proof. It suffices to prove that every (Pro-U)-sink $\{\underline{f}^\alpha: \underline{UK}^\alpha \to \underline{S} \mid \alpha\varepsilon\Lambda\}$, $\underline{S}=(S_a, s_{ab}, A)$ ε Pro-SET, has a unique (Pro-U)-final lifting; that is there exists a unique, up to isomorphisms, $\underline{H}$ ε Pro-C and a(Pro-U)-final sink $\{\underline{g}^\alpha: \underline{K}^\alpha \to \underline{H} \mid \alpha\varepsilon\Lambda\}$, such that $\underline{S} = \underline{UH}$ and $\underline{g}^\alpha = U\underline{f}^\alpha$, for all $\alpha\varepsilon\Lambda$.
We only sketch the construction of $\underline{H}$, leaving all other easy details to the reader. For every $a\varepsilon A$, let $\{f^\alpha_{\phi(a)}: UK^\alpha_{\phi(a)} \to S_a \mid \alpha\varepsilon\Lambda\}$ be the sink where $f^\alpha_{\phi(a)}$ is a representative of $\underline{f}^\alpha_a : \underline{UK}^\alpha \to S_a$, and let $\{g^\alpha_{\phi(a)}: K^\alpha_{\phi(a)} \to H_a \mid \alpha\varepsilon\Lambda\}$ be its U-final lifting, which there exists by hypothesis. Then, by the properties of the final lifting, it follows that, for all $a \leq b$, $s_{ab}: S_b \to S_a$ comes from a certain C-morphism $q_{ab}: H_b \to H_a$, so that $\underline{H} = (H_a, q_{ab}, A)$ is an inverse system in C.

By virtue of the above theorem, since topological functors (1.9) preserve monomorphisms and epimorphisms, all the results in this section, characterizing monomorphisms and epimorphisms in Pro-SET, can be extended to Pro-C as well, where C is any topological category over SET.

REMARKS.

2.10. Let $\underline{K}$ be an inverse system in HComp, the category of compact Hausdorff spaces, and let $\underline{p}$: $K \to \underline{K}$ be the inverse limit morphism [1] . If $p_i(K)$ is an open set in K_i, for all i, then $\underline{p}$ is an epimorphism in Pro-HComp. This follows from [2] , Th.3.7, p.217, and Th. 2.6. above.

2.11. Let HLcomp be the category of locally compact Hausdorff spaces and let TYCH be the category of completely regular T_1 spaces. Every $X \in$ TYCH admits an HLcomp-expansion [3,11] $\underline{p}$: $X \to \underline{K}$, where $\underline{K} \in$ Pro-HLcomp is formed by taking all open neighbourhoods of X in βX, its Stone-Čech compactification, directed by reversed inclusion. $\underline{p}$ is an epimorphism in Pro-TYCH, in fact each p_i is an epimorphism in TYCH.
If S is any topological space and η: $S \to X$ is its epireflection in TYCH, then $\underline{p} \cdot \eta$: $S \to X \to \underline{K}$ is an HLcomp-expansion of S which is not an epimorphism in Pro-TOP (TOP being the category of all topological spaces and continuous maps) This may be seen using Th. 2.6.

3. GENERAL MONOMORPHISMS AND EPIMORPHISMS IN Pro-$\mathcal{C}$. Let now $\mathcal{C}$ be an arbitrary category.

LEMMA 3.1. Let $\underline{e}$: $\underline{H} \to \underline{K}$, $\underline{f}$, $\underline{g}$: $\underline{K} \to R$ be given morphisms in Pro-$\mathcal{C}$. Then $\underline{f}\cdot\underline{e} = \underline{g}\cdot\underline{e}$ holds iff there is an index $i \in I$ such that $f_i \cdot \underline{e}_i = g_i \cdot \underline{e}_i$, where f_i, g_i represent $\underline{f}$, $\underline{g}$, respectively.

Proof. Follows from the definitions and from 1.5.

PROPOSITION 3.2. $\underline{e}$: $\underline{H} \to \underline{K}$ is an epimorphism in Pro-$\mathcal{C}$ iff there is an index $i \in I$ such that $\underline{e}_i$: $\underline{H} \to K_i$ is an epimorphism, i.e. has a representing epicofinal sink.

Proof. Let $\underline{e}_i$ be an epimorphism. If $\underline{f}$, $\underline{g}$: $\underline{K} \to \underline{R}$ are such that $\underline{f}\cdot\underline{e} = \underline{g}\cdot\underline{e}$, then $\underline{f}_c \cdot \underline{e} = \underline{g}_c \cdot \underline{e}$, for all $c \in C$, which means $f_{ic} \cdot \underline{e}_i = g_{ic} \cdot \underline{e}_i$, by the Lemma, hence $f_{ic} = g_{ic}$, for all $c \in C$. It follows $\underline{f} = \underline{g}$, and $\underline{e}$ is an epimorphism.
Conversely, let $\underline{e}$ be an epimorphism and suppose that no $\underline{e}_i$, $i \in I$, is an epimorphism. Then, for all $i \in I$, there are $\underline{f} \neq \underline{g}$: $K_i \to \underline{R}$ such that $\underline{f}\cdot\underline{e}_i = \underline{g}\cdot\underline{e}_i$. This last implies $f_c \cdot \underline{e}_i = g_c \cdot \underline{e}_i$, for all $c \in C$, then $\underline{f}_c \cdot \underline{e} = \underline{g}_c \cdot \underline{e}$, for all $c \in C$, by the Lemma. By assumption one obtains $\underline{f}_c = \underline{g}_c$, for all $c \in C$, that is $\underline{f} = \underline{g}$, which is a contradiction.

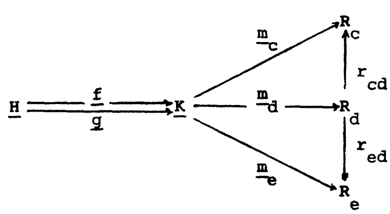
PROPOSITION 3.3. If $\underline{m}: \underline{K} \to \underline{R}$ is such that $\underline{m}_c: \underline{K} \to R_c$ is a monomorphism, for some $c \in C$, then also $\underline{m}$ is a monomorphism in Pro-$\mathcal{C}$.

Proof. Suppose $\underline{m}_c$ is a monomorphism in Pro-$\mathcal{C}$and let $\underline{f},\underline{g}: \underline{H} \to \underline{K}$ be such that $\underline{m}\cdot\underline{f} = \underline{m}\cdot\underline{g}$. This means $\underline{m}_d\cdot\underline{f} = \underline{m}_d\cdot\underline{g}$, for all $d \in C$, then $\underline{f} = \underline{g}$.

This proposition may be inverted in some cases, such as the following.

PROPOSITION 3.4. Let $\underline{m}: \underline{K} \to \underline{R}$ be a monomorphism in Pro-$\mathcal{C}$. If there is an index $c \in C$ with the property that $r_{cd}: R_d \to R_c$ is a monomorphism in $\mathcal{C}$ for every $d \geq c$, then $\underline{m}_c: \underline{K} \to R_c$ is a monomorphism in Pro-$\mathcal{C}$.

Proof. Let $\underline{f}$, $\underline{g}$: $\underline{H} \to \underline{K}$ be such that $\underline{m}_c\cdot\underline{f} = \underline{m}_c\cdot\underline{g}$. Then, for all $e \in C$, one has the following commutative diagram, where $d \geq c,e$,



Since r_{cd} is a monomorphism, $\underline{m}_e\cdot\underline{f} = \underline{m}_e\cdot\underline{g}$. Finally $\underline{f} = \underline{g}$ by the assumption and by 1.5.

Note that, in case $\mathcal{C}$ = SET or $\mathcal{C}$ is a topological category over SET, then one can apply the results of section 2 in order to obtain information about monomorphisms and epimorphisms in Pro-$\mathcal{C}$ of the general form $\underline{H} \to \underline{K}$.

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