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ZERO-DIMENSIONAL OPEN MAPPINGS  
WHICH INCREASE DIMENSION  
A. CHIGOGIDZE

**Abstract:** Using a notion of 0-soft rectangular diagram it is shown that any positive-dimensional compact of weight  $\tau$  is an image of a 1-dimensional compact of the same weight under 0-soft mapping whose fibers are all homeomorphic to the Cantor cube  $D^\tau$  of weight  $\tau$ .

**Key words:** AB(0), 0-soft mapping, 0-soft diagram.

**Classification:** 54F45, 54C10, 54C55

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**Introduction.** It is well-known that open mappings with zero-dimensional fibers can arbitrarily increase dimension. The first example of such mapping was described by Kolmogorov in [7]. In this example the domain is a 1-dimensional Peano continuum and the range is 2-dimensional. Then Keldyš [6] constructed a similar example where the range is a square  $I^2$ . The fact that Menger's universal curve can be mapped onto every Peano continuum by an open mapping whose fibers are all homeomorphic to the Cantor set was established by Wilson [12]. The most general result in this direction is the following important theorem of Pasynkov [10]:

**Pasynkov's Theorem.** Any positive-dimensional compact of weight  $\tau$  is an image of a 1-dimensional (in the sense of dim) compact of the same weight under an open mapping with zero-dimensional fibers.

The purpose of this note is to prove a stronger result than the last one. Namely, the following:

Theorem. Any positive-dimensional compact of weight  $\tau$  is an image of a 1-dimensional compact of the same weight under a 0-soft mapping whose fibers are all homeomorphic to the Cantor cube  $D^\tau$  of weight  $\tau$ .

Before getting down to the proof of this theorem, I'll recall some preliminary notions and results that will be needed.

Some preliminary notions and results. All the topological spaces considered will be compact (and Hausdorff), and the mappings - continuous. Under the dimension I mean the covering dimension  $\dim$ .  $wX$  denotes the weight of a space  $X$ .

Recall that a compact  $X$  is an absolute extensor for zero-dimensional compacts ( $X \in AE(0)$ ), if for any zero-dimensional compact  $B$  and for any its closed subset  $A$  any mapping  $g:A \rightarrow X$  can be extended over the whole  $B$ . Such spaces have been actively studied (see a recent survey of Ščepin [11]; note here that dimensional properties of  $AE(0)$ 's have been considered also in [4],[2]). A similar property of a mapping is 0-softness. Let us recall that a mapping  $f:X \rightarrow Y$  is said to be 0-soft if for any zero-dimensional compact  $B$ , any closed subset  $A$  of it, and any two mappings  $g:A \rightarrow X$  and  $h:B \rightarrow Y$  such that  $fg = h/A$  there exists a mapping  $k:B \rightarrow X$  such that  $fk = h$  and  $k/A = g$ .

From Ščepin's results it follows that every 0-soft mapping is open and surjective. Michael's selection theorem [9] implies that for mappings between metric compacts the converse is also true. On the other hand it is easy to see that a constant

mapping of a compact  $X$  is 0-soft iff  $X$  is  $AE(0)$ . So a simple example of an open but not 0-soft mapping is a constant mapping of any compact which is not  $AE(0)$ . Thus the class of 0-soft mappings is strictly included in the class of open ones.

An inverse system  $\{X_\alpha, p_\alpha^\beta, \lambda\}$  indexed by the ordinals less than some  $\lambda$  is said to be continuous if for all limit ordinals  $\gamma < \lambda$ , the natural mapping from  $X_\gamma$  to  $\lim\{X_\alpha, p_\alpha^\beta, \gamma\}$  is a homeomorphism. If  $S_1 = \{X_\alpha, p_\alpha^\beta, \lambda\}$  and  $S_2 = \{Y_\alpha, q_\alpha^\beta, \lambda\}$  are inverse systems and for each  $\alpha < \lambda$   $f_\alpha : X_\alpha \rightarrow Y_\alpha$  is a mapping such that all the rectangular diagrams that arise are commutative, then the system  $\{f_\alpha : \alpha < \lambda\}$  is called a morphism between  $S_1$  and  $S_2$ . Clearly, in this case, there exists a limit mapping  $\lim f_\alpha : \lim S_1 \rightarrow \lim S_2$ . I'll denote rectangular diagrams consisting of the compacts  $X_{\alpha+1}, Y_{\alpha+1}, X_\alpha, Y_\alpha$  and corresponding mappings  $f_{\alpha+1}, f_\alpha, p_\alpha^{\alpha+1}, q_\alpha^{\alpha+1}$  by  $D(\alpha, \alpha+1)$ .

Let us recall [11] also that for a commutative diagram

$$D = \begin{array}{ccc} X_2 & \xrightarrow{f_2} & Y_2 \\ \downarrow p & & \downarrow q \\ X_1 & \xrightarrow{f_1} & Y_1 \end{array}$$

the diagonal  $p \Delta f_2$  (see [3]) of the mappings  $p$  and  $f_2$  considered as a mapping of  $X_2$  into the fibered product of  $X_1$  and  $Y_2$  with respect to  $f_1$  and  $q$  (see [1]) is said to be a characteristic mapping of  $D$ .

Clearly, a diagram "is" a corresponding fibered product iff its characteristic mapping is a homeomorphism. Note also that a diagram is bicommulative [8] iff its characteristic

mapping is surjective (see, for example, [11]).

0-soft diagrams and the proof of the theorem.

Definition. A rectangular commutative diagram is said to be 0-soft iff its characteristic mapping is 0-soft.

Lemma 1. Let  $D$  be a 0-soft diagram. Then for any zero-dimensional compact  $B$ , any closed subset  $A$  of it, and any three mappings  $h: B \rightarrow Y_2$ ,  $g: A \rightarrow X_2$  and  $k_1: B \rightarrow X_1$  such that  $f_1 k_1 = qh$ ,  $f_2 g = h/A$  and  $pg = k_1/A$  there exists a mapping  $k_2: B \rightarrow X_2$  such that  $g = k_2/A$ ,  $f_2 k_2 = h$  and  $pk_2 = k_1$ .

Proof. Let  $Z$  be a fibered product of  $X_1$  and  $Y_2$  with respect to  $f_1$  and  $q$ ; let  $q': Z \rightarrow X_1$  and  $f_1': Z \rightarrow Y_2$  be the canonical projections (see [1]). Clearly the diagonal  $k_1 \Delta h$  maps  $B$  into  $Z$ . It is not hard to show that  $(k_1 \Delta h)/A = rg$  where  $r: X_2 \rightarrow Z$  is a characteristic mapping of  $D$ . By our assumption  $r$  is 0-soft. Consequently, there exists a mapping  $k_2: B \rightarrow X_2$  such that  $rk_2 = k_1 \Delta h$  and  $g = k_2/A$ . It is easy to check that  $k_2$  is the desired mapping. The lemma is thereby proved.

Lemma 2. Let  $S_1 = \{X_\alpha, p_\alpha^\beta, \lambda\}$  and  $S_2 = \{Y_\alpha, q_\alpha^\beta, \lambda\}$  be two well-ordered continuous inverse systems. Suppose that  $\{f_\alpha, \alpha < \lambda\}$  is a morphism between  $S_1$  and  $S_2$  such that for each  $\alpha < \lambda$  ( $\lambda$  is a limit number) diagrams  $D(\alpha, \alpha + 1)$  are 0-soft. Then the limit mapping  $f = \lim f_\alpha: \lim S_1 \rightarrow \lim S_2$  is 0-soft whenever  $f_0$  is 0-soft mapping.

Proof. Suppose that  $B$  is a zero-dimensional compact.  $A$  is a closed subset of  $B$  and  $g: A \rightarrow \lim S_1$  and  $h: B \rightarrow \lim S_2$  are mappings such that  $fg = h/A$ . Let us prove that there is an extension  $k: B \rightarrow \lim S_1$  of  $g$  such that  $fk = h$ . Set

$g_\alpha = p_\alpha \cdot g$  and  $h_\alpha = q_\alpha \cdot h$  ( $\alpha < \lambda$ ) where  $p_\alpha$  and  $q_\alpha$  are the limit projections of  $S_1$  and  $S_2$  respectively. For  $\alpha = 0$  the mapping  $f_0: X_0 \rightarrow Y_0$  is 0-soft by assumption. Consequently, there exists an extension  $k_0: B \rightarrow X_0$  of the mapping  $g_0: A \rightarrow X_0$  such that  $f_0 k_0 = h_0$ . Suppose now that for all  $\alpha < \beta$  ( $\beta < \lambda$ ) we have already constructed extensions  $k_\alpha: B \rightarrow X_\alpha$  of the mappings  $g_\alpha$  in such a way that all the diagrams that arise are commutative (in particular,  $f_\alpha k_\alpha = h_\alpha$ ). If the number  $\beta$  is a limit ordinal, then we set  $k_\beta = \lim_{\alpha < \beta} k_\alpha: B \rightarrow X_\beta$ . Obviously,  $k_\beta / A = g_\beta$ ,  $f_\beta k_\beta = h_\beta$  and  $p_\alpha^\beta k_\beta = k_\alpha$  ( $\alpha < \beta$ ).

Suppose now that  $\beta = \alpha + 1$ . By assumption, the diagram  $D(\alpha, \alpha + 1)$  is 0-soft. Consequently, we can use lemma 1 and so, there exists an extension  $k_{\alpha+1}: B \rightarrow X_{\alpha+1}$  of  $g_{\alpha+1}$  such that  $f_{\alpha+1} k_{\alpha+1} = h_{\alpha+1}$  and  $p_\alpha^{\alpha+1} k_{\alpha+1} = k_\alpha$ .

Continuing the construction, we get a family  $\{k_\alpha\}$  of extensions of the mappings  $g_\alpha$ ,  $\alpha < \lambda$ , and all the diagrams arising are commutative. Next, let  $k = \lim k_\alpha: B \rightarrow \lim S_1$ . It is easy to check that  $k$  is the desired extension of  $g$ . Thus  $f$  is 0-soft. The lemma is proved.

**Lemma 3.** Any positive-dimensional compact  $Y$  of weight  $\tau$  is an image of a 1-dimensional compact  $X$  of the same weight  $\tau$  under 0-soft mapping with zero-dimensional fibers.

**Proof** (by induction on the weight of  $Y$ ). Suppose that  $WY = \aleph_0$ , i.e.,  $Y$  is metrizable. It follows from Pasynkov's theorem mentioned in the introduction that  $Y$  is an image of a 1-dimensional metrizable compact under an open mapping with zero-dimensional fibers. But every open mapping between metrizable compacts is 0-soft.

Suppose now that the lemma has been proved for all positive-dimensional compact spaces  $Y$  of weight  $< \tau$ , and let  $wY = \tau$ . As it is well-known, we can represent  $Y$  as a limit space of a continuous well-ordered inverse system  $S_Y = \{Y_\alpha, q_\alpha^\beta, \lambda\}$  where all  $Y_\alpha$ 's are compacts with  $wY_\alpha < \tau$ . Without loss of generality we can assume that each  $Y_\alpha$  is positive-dimensional.

Now let us construct a continuous well-ordered inverse system  $S_X = \{X_\alpha, p_\alpha^\beta, \lambda\}$  "parallel" to  $S_Y$  and a morphism  $\{f_\alpha\}: S_X \rightarrow S_Y$  such that  $\lim S_X$  and  $\lim f_\alpha$  will be the desired objects.

By the inductive hypothesis  $Y_0$  is an image of a 1-dimensional compact  $X_0$  with  $wX_0 = wY_0$  under a 0-soft mapping  $f_0$  with zero-dimensional fibers. Suppose now that for all  $\alpha < \gamma$  ( $\gamma < \lambda$ ) we have already constructed:

- a) 1-dimensional compacts  $X_\alpha$  with  $wX_\alpha = wY_\alpha$ ;
- b) 0-soft mappings  $f_\alpha: X_\alpha \rightarrow Y_\alpha$  with zero-dimensional fibers;
- c) mappings  $p_\alpha^\beta: X_\beta \rightarrow X_\alpha$  ( $\alpha < \beta < \gamma$ )

such that

- 1) all the diagrams  $D(\alpha, \alpha+1)$  are 0-soft ( $\alpha < \gamma$ );
- 2)  $p_\alpha^\beta = p_\alpha^\sigma \cdot p_\sigma^\beta$  for every  $\alpha, \sigma$  and  $\beta$  with  $\alpha < \sigma < \beta < \gamma$ .

Let us construct  $X_\gamma$ ,  $f_\gamma$  and  $p_\alpha^\gamma$  ( $\alpha < \gamma$ ).

If the number  $\gamma$  is a limit ordinal, then we set  $X_\gamma = \lim \{X_\alpha, p_\alpha^\beta, \gamma\}$  and  $f_\gamma = \lim_{\alpha < \gamma} f_\alpha$ . Obviously,  $\dim X_\gamma = 1$ ,  $wX_\gamma = wY_\gamma$ . It is easy to check (see [1]) that  $f_\gamma$  has a zero-dimensional fibers. By lemma 2,  $f_\gamma$  is 0-soft. Clearly,  $p_\alpha^\gamma$  are the limit projections of an inverse system  $\{X_\alpha, p_\alpha^\beta, \gamma\}$ .

Suppose now that  $\gamma = \alpha + 1$ .

Consider a fibered product  $Z$  of  $X_\alpha$  and  $Y_{\alpha+1}$  with respect

to  $f_\alpha$  and  $q_\alpha^{\alpha+1}$ . Let  $(q_\alpha^{\alpha+1})': Z \rightarrow X_\alpha$  and  $f_\alpha': Z \rightarrow Y_{\alpha+1}$  be the canonical projections. It follows from [1] that  $f_\alpha'$  is a surjection with a zero-dimensional fiber. It is easy to see also that  $f_\alpha'$  is 0-soft, in particular, open and hence  $\dim Z > 0$  (recall that  $Y_{\alpha+1}$  is positive-dimensional). Since all the mappings considered are perfect,  $wZ \geq wY_{\alpha+1}$ . On the other hand, since  $Z$  is a subspace of the topological product  $X_\alpha \times Y_{\alpha+1}$ , we have the following inequalities:  $wZ \leq \max\{wX_\alpha, wY_{\alpha+1}\} = \max\{wY_\alpha, wY_{\alpha+1}\} \leq wY_{\alpha+1}$ . Thus  $wZ = wY_{\alpha+1} < \tau$ . Again by the inductive hypothesis  $Z$  is an image of a 1-dimensional compact  $X_{\alpha+1}$  with  $wX_{\alpha+1} = wZ = wY_{\alpha+1}$  under 0-soft mapping  $h$  with zero-dimensional fibers. Set  $f_{\alpha+1}' = f_\alpha' \cdot h$  and  $p_\alpha^{\alpha+1} = (q_\alpha^{\alpha+1})' \cdot h$ . Obviously,  $f_{\alpha+1}'$  is 0-soft as a composition of 0-soft mappings and has zero-dimensional fibers as a composition of mappings with zero-dimensional fibers. Finally, let us note that the arised rectangular diagram is 0-soft since its characteristic mapping, as it is easy to see, coincides with 0-soft mapping  $h$ .

Continuing the construction, we set  $X = \lim\{X_\alpha, p_\alpha^\beta, \lambda\}$  and  $f = \lim f_\alpha$ . Obviously,  $\dim X = 1$ ,  $wX = wY$  and  $f$  is surjective mapping with zero-dimensional fibers. By lemma 2,  $f$  is 0-soft. The lemma is thereby proved.

Now I give a proof of the theorem stated in the introduction.

**Proof.** Let  $Y$  be a positive-dimensional compact of weight  $\tau$ . By lemma 3,  $Y$  is an image of a 1-dimensional compact  $Z$  with  $wZ = \tau$  under 0-soft mapping  $h$  with zero-dimensional fibers. Set  $X = Z \times D^\tau$ , where  $D^\tau$  is the Cantor cube of weight  $\tau$ . Let  $p: X \rightarrow Z$  be a natural projection. Since  $D^\tau$  is  $AE(0)$ , it is easy to check (see, for example, [5]) that  $p$  is 0-soft.



Consequently, a composition  $f = hp$  is also 0-soft. For each point  $y$  from  $Y$  its inverse image  $f^{-1}(y)$  coincides with the product  $h^{-1}(y) \times D^\tau$ . Since  $h^{-1}(y)$  is a zero-dimensional AN(0) of weight  $\leq \tau$  we can conclude, using a characterization of  $D^\tau$  given by Štěpán (see [11], theorem 1), that  $f^{-1}(y)$  is homeomorphic to  $D^\tau$ . To complete the proof, we only have to remark that  $\dim X = 1$  and  $wX = \tau$ .

#### R e f e r e n c e s

- [1] P.S. ALEXANDROV, B.A. PASYNKOV: Introduction to Dimension Theory (Nauka, Moscow, 1973).
- [2] A. CHIGOGIDZE: On a generalization of perfectly normal spaces, Topology Appl. 13(1982), 15-20.
- [3] R. ENGELKING: General Topology (PWN, Warszawa, 1977).
- [4] V.V. FEDORČUK: On the dimension of  $k$ -metrizable bicompacta, in particular Dugundji spaces, Dokl. Akad. Nauk SSSR 234(1977), 30-33.
- [5] B. HOFFMAN: Locally Dugundji spaces and quotient spaces of locally compact groups, Quart. J. Math. 31 (1980), 181-190.
- [6] L. KELDYŠ: An example of a one-dimensional continuum which can be mapped onto a square by a zero-dimensional open mapping, Dokl. Akad. Nauk SSSR 97(1954), 201-204.
- [7] A. KOLMOGOROV: Über offene Abbildungen, Ann. of Math. 38(1937), 36-38.
- [8] K. KURATOWSKI: Topology, vol. 1 (Academic Press, New York, 1966).
- [9] E. MICHAEL: Selected selection theorems, Amer. Math. Monthly 63(1956), 233-238.
- [10] B.A. PASYNKOV: Zero-dimensional open mappings which arise dimension, Uspehi Mat. Nauk 18(1963), 183-190.

- [11] E.V. ŠČEPIN: Functors and uncountable powers of compacta, *Uspehi Mat. Nauk* 36(1981), 3-62.
- [12] D.C. WILSON: Open mappings of the universal curve onto continuous curves, *Trans. Amer. Math. Soc.* 168(1972), 497-515.

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