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ON MAXIMUM PRINCIPLES AND LIOUVILLE THEOREMS FOR QUASILINEAR ELLIPTIC EQUATIONS AND SYSTEMS

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Abstract: In the study of elliptic systems of partial differential equations it is customary to prove maximum principles for the modulus $|u|$ of a vector valued solution $\vec{u}$ and Liouville theorems for the vector $\vec{u}$. This note contains maximum principles for u and a Liouville theorem for $|\vec{u}|$.

Key words: Quasilinear elliptic equation, maximum principle.

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The classical maximum principle states the following: If $u: \Omega \longrightarrow \mathbb{R}$ is a classical solution of an elliptic second order differential equation, then u attains its maximum on the boundary $\partial\Omega$. The strong version of the maximum principle even reads: If u attains its maximum in the interior of Ω, then u is constant.

The classical Liouville Theorem says: If $u: \mathbb{R}^n \longrightarrow \mathbb{R}$ is a bounded solution of an elliptic second order differential equation, then u is constant.

The validity of these theorems is well known for linear and quasilinear differential equations [4,5,6,8,11].

For elliptic systems the natural question arises which of the two quantities $|\vec{u}|$ or $\vec{u}$ serves as an appropriate generalization of u. To the author's knowledge, up to now classical maximum principles have been examined for $|u|$ [5,14] and Liouville theorems for the vector function $\vec{u}$ [6,11,12].

This note contains, among other results:

a) a strong maximum principle for $|\vec{u}|$,

b) a maximum principle for the components of $\vec{u}$, and

c) a Liouville theorem for $|\vec{u}|$ under weaker assumptions than the ones which guarantee such a theorem for $\vec{u}$.

Our investigations were motivated by the following examples.

<u>Example 1</u> [11]. Let $x \in \mathbb{R}^n$, $u^1(x) = \sin x_1$, $u^2(x) = \cos x_1$. The vector $u(x) = (u^1(x), u^2(x))$ is a bounded, real analytic solution of the system

$$-\Delta u^i = \frac{u^i}{\max\{1, |\nabla u|\}} |\nabla u|^2, \; i = 1,2, \text{ on } \mathbb{R}^n.$$

<u>Example 2</u> [6]: Let $x \in \mathbb{R}^n$ and $u^i(x) = \frac{x_i}{|x|}$ for $i = 1,\ldots,n$; $n \geq 3$. Then the vector $u(x)$ is a bounded weak solution of the system

$$-\Delta u^i = \frac{2u^i}{1 + |u|^2} |\nabla u|^2, \; i = 1,\ldots,n, \text{ on } \mathbb{R}^n.$$

<u>Example 3</u> [6]: Let $x \in \mathbb{R}^n$ and $u^i(x) = x_i (1+|x|^2)^{-1/2}$, $i = 1,\ldots,n$; $n \geq 3$. Then the vector $u(x)$ is a real analytic bounded solution of a system of type

$$-\Delta u^i = u^i g(x, \nabla u), \; i = 1,\ldots,n, \text{ on } \mathbb{R}^n,$$

with $|u| \, |g(x,u)| \leq |\nabla u|^2$.

We use the following assumptions and notations. Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a (not necessarily bounded, but open) domain with Lipschitz continuous boundary $\partial\Omega$. We consider a system of uniformly strongly elliptic quasilinear differential equations in Ω

$$-\sum_{i,j=1}^{n} (A_{ij}(x,u,Du)) u^i_{x_i})_{x_j} = f^k(x,u,Du), \; k = 1,\ldots,N.$$

In shorthand we shall write it as

$$(1) \qquad -\mathrm{div}\,(A(x,u,Du)\,\nabla u^k) = f^k(x,u,Du).$$

We suppose that system (1) has a weak bounded solution, i.e., a vector function $u = (u^1,\dots,u^N)$ with components $u^k \in H^{1,2}_{loc}(\Omega) \cap \cap L^\infty(\Omega)$, such that the following relation holds for any test vector $\Phi \in [C_0^\infty(\Omega)]^N$.

$$\int_\Omega A(x,u,Du)\,\nabla u^k\ \nabla\Phi^k\,dx = \int_\Omega f^k(x,u,Du)\,\Phi^k(x)\,dk, \quad k = 1,\dots,N.$$

For the question of existence and regularity we refer to [1,2, 5,11]. The $L^\infty(\Omega)$-norm of u is denoted by M. The coefficients $A_{ij}(x,u,p)$ are assumed to be symmetric, i.e.,

$$A_{ij}(x,u,p) = A_{ji}(x,u,p) \text{ for } i,j = 1,\dots,n \text{ and } x,u,p \in \Omega,\ B_M(0),\ \mathbb{R}^{nN},$$

and to be bounded. Furthermore, we require the ellipticity condition that there exist $\lambda,\ \mu \in \mathbb{R}^+$ such that

$$\lambda\,|\xi|^2 \le \sum_{i,j=1}^{n} A_{ij}(x,u,p)\,\xi_i\,\xi_j \le \mu\,|\xi|^2 \text{ for } x,u,p,\xi \in \Omega,\ B_M(0),\ \mathbb{R}^{nN},\ \mathbb{R}^n.$$

Finally the functions $A_{ij}(x,u,p)$ and $f^k(x,u,p)$ are assumed to be measurable in $x \in \Omega$ and continuous in $u \in B_M(0)$ and $p \in \mathbb{R}^{nN}$ for $i,j = 1,\dots,n$ and $k = 1,\dots,N$.

From time to time we shall make one or more of the following assumptions:

(A1) The right hand side $f = (f^1,\dots,f^N)$ has "quadratic growth in p", i.e. there exists a positive number a such that

$$|f(x,u,p)| \le a|p|^2 \text{ for } x,u,p \in \Omega,\ B_M(0),\ \mathbb{R}^{nN}$$

(A2) There exists a real number $\lambda^* \le \lambda$ such that

$$u\,f(x,u,p) \le \lambda^*\,|p|^2 \text{ for } x,u,p \in \Omega,\ B_M(0),\ \mathbb{R}^{nN}.$$

(A3) <u>The coefficients</u> $A_{ij}(x,u,p)$ <u>are Lipschitz continuous in their arguments.</u>

Notice that the common assumption $\lambda^* < \lambda$, which is known to be optimal in a different context, is weakened in (A2). Let us also point out that we do not require any smallness condition e.g., of type $a M < \lambda$.

<u>Maximum principles.</u> Our first result shall be a maximum principle for the modulus of u. Therefore we apply the principal part of system (1) to the function $|u|^2$ and obtain

$$(2)\quad -\mathrm{div}(A(x,u,Du)\nabla(|u|^2)) \le \sum_{k=1}^{N} (2u^k f^k(x,u,Du) - 2\lambda|\nabla u^k|^2)$$

$$\le \begin{cases} 2(\lambda^* - \lambda)\,|\nabla u|^2 \le 0, \text{ if (A2) holds,} \\ 2 \text{ sign } u\; Du\; D(u^2), \text{ if } N=1 \text{ and (A1) holds.} \end{cases}$$

In both cases $|u|^2$ solves an elliptic differential inequality and therefore the following theorems are a consequence of maximum principles for differential inequalities.

<u>Theorem 1.</u> Let u <u>be a bounded weak solution of system (1). Suppose that</u> Ω <u>is bounded and that (A2) holds. Then the following maximum principle holds:</u>

i) $\sup_{\Omega} |u| \le \sup_{\partial\Omega} |u|$, <u>where sup means the essential suprenorm. Furthermore, if</u> $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ <u>and if (A3) holds, then the strong maximum principle holds, i.e.</u>

ii) $|u|$ <u>is constant provided</u> $|u|$ <u>attains its maximum in</u> Ω.

<u>Theorem 2.</u> <u>Suppose</u> N=1, <u>assumption (A3) holds and</u> $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ <u>is a solution of the differential inequality.</u>

$$|\mathrm{div}(A(x,u,Du)\, Du| \le a\, |Du|^2$$

<u>Then the strong maximum principle holds</u>, i.e. $|u|$ <u>is constant</u>

if $|u|$ *attains its maximum in* Ω.

Proof: Theorem 1i) follows from Stampacchia's maximum principle [9, p. 39]. Theorem 1ii) and Theorem 2 are a consequence of Hopf's strong maximum principle [4, p. 34]. For Theorem 2 one has to interpret the right hand side of (2) as a linear term in $D(|u|^2)$ with locally bounded vector-coefficient a sign u Du.

Remark: Theorem 1 applies in particular to the examples above. Furthermore, Theorem 1 implies that the homogeneous Dirichlet problem for system (1) has only the trivial solution $u \equiv 0$ in $[H_0^{1,2}(\Omega) \cap L^\infty(\Omega)]^N$.

Let us now try to prove maximum principles for the components u^k of u. This will not be possible without suitable assumptions on the structure of the right hand sides of system (1). Since we shall concentrate on an arbitrary but fixed component u^k in the sequel, let $k \in \{1,\dots,N\}$ be fixed. As the situation demands we shall require one of the following assumptions.

(A4) *There exists a number* $a' \in \mathbb{R}^+$ *such that*

$$|f^k(x,u,p)| \leq a'|p_k|^2 \text{ for } x,u,p \in \Omega,\ B_M(0),\ \mathbb{R}^{nN}.$$

Notice that the requirement "(A4) holds for every $k=1,\dots,N$" is considerably stronger than (A1). This is why Theorem 3i) contains an apparently stronger result. Other suitable assumptions are sign-conditions on f^k:

(A5) $\operatorname{sign} u^k \operatorname{sign} f^k \leq 0$ for $x,u,p \in \Omega,\ B_M(0),\ \mathbb{R}^{nN}$,

or

(A6) $\operatorname{sign} u^k \operatorname{sign} f^k \geq 0$ for $x,u,p \in \Omega,\ B_M(0),\ \mathbb{R}^{nN}$.

Assumption (A6) is easily verified for examples 1 and 2. Now we can formulate the following results concerning the component u^k.

Theorem 3.

i) Let u be a bounded weak solution of system (1) on a bounded domain Ω. Under assumptions (A3) and (A4) the component u^k attains its maximal modulus on $\partial\Omega$, i.e. $\sup_{\partial\Omega} |u^k| = \sup_{\Omega} |u^k|$. If furthermore $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$, then the strong maximum principle holds for u^k: u^k is constant in Ω, if $|u^k|$ attains its maximum in Ω.

ii) Part i) remains valid if (A4) is replaced by (A5).

iii) Let $u^k \in C^2(\Omega) \cap C^0(\bar{\Omega})$ be a nonnegative component of a solution to system (1) on a bounded domain Ω and suppose that (A3) and the sign-condition (A6) holds. Then u^k attains its minimum on $\partial\Omega$ and the strong minimum principle holds for u^k: u^k is constant in Ω, if u_k attains its minimum in Ω.

Proof: Part i) follows from the observation that u^k is a solution of the differential inequality $|\mathrm{div}(A(x,u,Du)\nabla u^k)| \leq \leq a^1 |\nabla u^k|^2$, and from Theorem 2. In case ii) we have $-\mathrm{div}(A(x,u,Du)\nabla((u^k)^2) \leq 0$, i.e. $|u^k|$ attains its maximum on $\partial\Omega$. For iii) one has to use the nonnegativity of u^k and assumption (A6) to conclude $-\mathrm{div}(A(x,u,Du)\nabla u^k)| \geq 0$, whence the desired result follows.

In view of the examples above, assumption (A6) appears to be sensible. If assumption (A6) holds for all components $k = = 1,\dots,N$ we have the following corollary.

Corollary 4: Let $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ be a solution of system (1) on a bounded domain Ω. Suppose (A3) holds as well as the sign condition (A6) for every $k=1,\dots,N$; and that none of the component u^k of u changes sign in Ω. Then for each $k=1,\ldots,N$ the function $|u^k|$ attains its minimum on the boundary, and the strong minimum principle holds: $|u^k|$ is constant in Ω, if $|u^k|$ attains its minimum in Ω.

Remarks: The nonnegativity of u^k, which was required in Theorem 3iii), can sometimes be verified. As it was kindly pointed out to the author by Professor J. Frehse, systems of type (1) occur in stochastic impulse control and there the sign of u^k i known to be positive. In general, however, it is hard to verify the assumption of Corollary 4 that no component of u changes sign. In spite of this apparent disadvantage, Corollary 4 has an interesting converse: If a nonconstant component u^k of a solution u to system (1) attains its minimum in Ω, then u^k has to change sign in Ω. This is indeed the case for the examples above.

Liouville theorems. Such theorems have been derived for solutions to nonlinear and quasilinear elliptic systems [3,6,7,8,10,11,12,13] and their importance lies in their close relation to the regularity question for solutions to system (1). The following theorem is a special case of a result by M. Meier [11] and can be interpreted as being analogous to Theorem 2.

Theorem 5 [11]: Let w be a weak bounded solution of the differential inequality (N=1)

$$(3)\qquad -\mathrm{div}(A(x,w,\nabla w)\,\nabla w) \le a\,|\nabla w|^2 \text{ in } \mathbb{R}^2.$$

Then w is constant.

As an immediate consequence we obtain a Liouville theorem for the modulus of the solution u of system (1), which parallels Theorem 1. All one has to do is to look at (2) again.

Corollary 6: Let u be a weak bounded solution of system (1) in $\mathbb{R}^2$ and suppose that (A2) holds. Then $|u|$ is constant.

Remarks: This is the case for example 1. Liouville theorems for the vector function u are stated elsewhere [3,6,8,10,11,12, 13], e.g. under assumption (A1) in $\mathbb{R}^n$ with $n \geq 2$ [11].

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