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Commentationes Mathematicae Universitatis Carolinae, Vol. 24 (1983), No. 4, 673--680

Persistent URL: <http://dml.cz/dmlcz/106265>

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THEOREMS ON MULTIFUNCTIONS SATISFYING
A RATIONAL INEQUALITY
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Abstract: We prove a fixed point theorem for a sequence of multifunctions satisfying a rational inequality which generalizes theorem 3 from [1].

Key words: Multifunction, fixed point.

Classification: 54H25

In [1] B. Fisher gave the following theorem

Theorem 1. Let S and T be mappings of the complete metric space into itself such that for all x, y in X either

$$(1) \quad d(Sx, Ty) \leq \frac{c \cdot d(x, Sx) \cdot d(y, Ty) + b \cdot d(x, Ty) \cdot d(y, Sx)}{d(x, Sx) + d(y, Ty)}$$

if $d(x, Sx) + d(y, Ty) \neq 0$ where $b \geq 0$ and $1 < c < 2$, or

$$d(Sx, Ty) = 0$$

otherwise. Then each of S and T has a fixed point and these points coincide.

We now prove a similar common fixed point theorem for two multifunctions T_1 and T_2 and for a sequence of multifunctions which generalize theorem 1.

The method used is a combination of methods used in [1]-[3].

Let (X, d) be a metric space. We denote by $CB(X)$ the set

of all nonempty closed bounded subsets of (X, d) and by H the Hausdorff-Pompeiu metric on $CB(X)$

$$H(A, B) = \max \left\{ \sup_{x \in A} d(x, B); \sup_{y \in B} d(y, A) \right\}$$

where $A, B \in CB(X)$ and

$$d(x, A) = \inf_{y \in A} d(x, y).$$

Let $A, B \in CB(X)$ and $k > 1$. In what follows, the following well-known fact will be used: For each $a \in A$, there is a $b \in B$ such that

$$d(a, b) \leq k H(A, B).$$

Let (X, d) be a metric space, we denote

$$\sigma(A, B) = \sup \{ d(a, b); a \in A \text{ and } b \in B \}$$

where $A, B \in CB(X)$. If A consists of a single point "a" we write

$$\sigma(A, B) = \sigma(a, B). \text{ If } \sigma(A, B) = 0 \text{ then } A = B = \{a\} \text{ (Lemma 1 [4]).}$$

Let $T: X \rightarrow X$ be a multifunction. Denote

$$F(T) = \{x \in X; x \in Tx\}.$$

Lemma. Let (X, d) be a metric space and $T_1, T_2: (X, d) \rightarrow CB(X)$ be two multifunctions. If

$$(2) \quad H^p(T_1x, T_2y) \leq \frac{c \cdot d(x, T_1x) \cdot d^p(y, T_2y) + b d(x, T_2y) \cdot d^p(y, T_1x)}{\sigma(x, T_1x) + \sigma(y, T_2y)}$$

holds for all $x, y \in X$ for which $\sigma(x, T_1x) + \sigma(y, T_2y) \neq 0$ where $p \geq 1$, $b \geq 0$ and $1 < c < 2$ and $F(T_1) \neq \emptyset$, then $F(T_2) \neq \emptyset$ and $F(T_1) = F(T_2)$.

Proof. Let $u \in F(T_1)$, then $u \in T_1u$ and if $d(u, T_2u) \neq 0$ then by (2) we have

$$\begin{aligned} d^p(u, T_2u) &\leq H^p(T_1u, T_2u) \leq \\ &\leq \frac{c \cdot d(u, T_1u) \cdot d^p(u, T_2u) + b d(u, T_2u) \cdot d^p(u, T_1u)}{\sigma(u, T_1u) + \sigma(u, T_2u)} \leq \end{aligned}$$

$$\leq \frac{c \cdot d(u, T_1 u) \cdot d^p(u, T_2 u) + b \cdot d(u, T_2 u) \cdot d^p(u, T_1 u)}{d(u, T_1 u) + d(u, T_2 u)}$$

which implies $d(u, T_2 u) = 0$. Since $T_2 u$ is closed, this shows that $u \in T_2 u$, which implies $F(T_1) \subset F(T_2)$. Analogously, $F(T_2) \subset F(T_1)$.

Theorem 2. Let (X, d) be a complete metric space and $T_1, T_2: X \rightarrow CB(X)$ two multifunctions such that for all $x, y \in X$ the inequality (2) holds if $\sigma(x, T_1 x) + \sigma(y, T_2 y) \neq 0$ where $p \geq 1, b \geq 0$ and $1 < c < 2$. Then T_1 and T_2 have common fixed points and $F(T_1) = F(T_2)$.

Proof. Choose a real number k with

$$(3) \quad 1 < k < \left(\frac{c}{b}\right)^{1/p}.$$

Let $x_0 \in X$ and $x_1 \in T_1 x_0$. Then there is an $x_2 \in T_2 x_1$ so that $d(x_1, x_2) \leq k H(T_1 x_0, T_2 x_1)$. Suppose $x_3, x_4, \dots, x_{2n-1}, x_{2n}, \dots$ could be chosen so that $x_{2n-1} \in T_1 x_{2n-2}, x_{2n} \in T_2 x_{2n-1}$ and

$$d(x_{2n-1}, x_{2n}) \leq k H(T_1 x_{2n-2}, T_2 x_{2n-1})$$

$$d(x_{2n-2}, x_{2n-1}) \leq k H(T_1 x_{2n-2}, T_2 x_{2n-3}).$$

Suppose first of all that

$$\sigma(x_{2n-2}, T_1 x_{2n-2}) + \sigma(x_{2n-1}, T_2 x_{2n-1}) = 0$$

for some n . Then $x_{2n-2} = \{T_1 x_{2n-2}\} = x_{2n-1} = \{T_2 x_{2n-1}\}$ and $x_{2n-2} = x_{2n-1}$ is a common fixed point for T_1 and T_2 .

Similarly $\sigma(x_{2n-1}, T_2 x_{2n-1}) + \sigma(x_{2n}, T_1 x_{2n}) = 0$ for some n implies that $x_{2n-1} = x_{2n}$ is a common fixed point for T_1 and T_2 .

Now suppose that $\sigma(x_{2n-2}, T_1 x_{2n-2}) + \sigma(x_{2n-1}, T_2 x_{2n-1}) \neq 0$ for $n=1, 2, \dots$. Then by (2) we have successively

$$d^p(x_{2n-1}, x_{2n}) \leq k^p H^p(T_1 x_{2n-2}, T_2 x_{2n-1}) \leq$$

$$\leq k^p \cdot \frac{cd(x_{2n-2}, T_1 x_{2n-2}) \cdot d^p(x_{2n-1}, T_2 x_{2n-1}) + bd(x_{2n-2}, T_2 x_{2n-1})}{(x_{2n-2}, T_1 x_{2n-2}) + (x_{2n-1}, T_2 x_{2n-1})} \\ \cdot \frac{d^p(x_{2n-1}, T_1 x_{2n-2})}{\dots} \leq \frac{k^p \cdot c \cdot d(x_{2n-2}, x_{2n-1}) \cdot d^p(x_{2n-1}, x_{2n})}{d(x_{2n-2}, x_{2n-1}) + d(x_{2n-1}, x_{2n})}$$

If $d(x_{2n-1}, x_{2n}) = 0$, then $x_{2n-1} = x_{2n}$ is a common fixed point for T_1 and T_2 . If $d(x_{2n-1}, x_{2n}) \neq 0$ then

$d(x_{2n-1}, x_{2n}) \leq (ck^p - 1) \cdot d(x_{2n-2}, x_{2n-1})$ for $n=0, 1, 2, \dots$. Similarly we have

$$d(x_{2n}, x_{2n+1}) \leq (ck^p - 1) \cdot d(x_{2n-1}, x_{2n}) \text{ for } n=0, 1, 2, \dots$$

Repeating the above argument, we obtained

$$d(x_n, x_{n+1}) \leq (ck^p - 1)^n d(x_0, x_1) \text{ for } n=0, 1, 2, \dots$$

Since $0 < (ck^p - 1) < 1$ by (3), then by routine calculation one can show that $\{x_n\}$ is a Cauchy sequence and since X is complete, we have $\lim x_n = u$ for some $u \in X$.

If we now suppose that $d(u, T_1 u) \neq 0$ then

$$d^p(x_{2n}, T_1 u) \leq H^p(T_2 x_{2n-1}, T_1 u) \leq \\ \leq \frac{cd(u, T_1 u) \cdot d^p(x_{2n-1}, T_2 x_{2n-1}) + bd(x_{2n-1}, T_1 u) \cdot d^p(u, T_2 x_{2n-1})}{\sigma(u, T_1 u) + \sigma(x_{2n-1}, T_2 x_{2n-1})} \leq \\ \leq \frac{cd(u, T_1 u) \cdot d^p(x_{2n-1}, x_{2n}) + bd(x_{2n-1}, T_1 u) \cdot d^p(u, x_{2n})}{d(u, T_1 u) + d(x_{2n-1}, x_{2n})}$$

and on letting n tend to infinity we have $d(u, T_1 u) \leq 0$. It follows that $d(u, T_1 u) = 0$. Since $T_1 u$ is closed, this shows that $u \in T_1 u$. By lemma $u \in T_2 u$ and $F(T_1) = F(T_2)$.

If $T_1 = T_2$ we have the following theorem:

Theorem 3. Let (X, d) be a complete metric space and let $T: (X, d) \rightarrow CB(X)$ be a multifunction such that

$$H^p(Tx, Ty) \leq \frac{cd(x, Tx).d(y, Ty)^p + bd(x, Ty).d(y, Tx)^p}{\delta(x, Tx) + \delta(y, Ty)}$$

holds for all $x, y \in X$ for which $\delta(x, Tx) + \delta(y, Ty) \neq 0$, where $p \geq 1$, $b \geq 0$ and $1 < c < 2$, then T has fixed points.

If T_1 and T_2 are single valued mappings we have the following theorem:

Theorem 4. Let T_1 and T_2 be mappings of a complete metric space (X, d) into itself such that for all x, y in X either

$$d^p(T_1x, T_2y) \leq \frac{cd(x, T_1x).d^p(y, T_2y) + b d(x, T_2y).d^p(y, T_1x)}{d(x, T_1x) + d(y, T_2y)}$$

if $d(x, T_1x) + d(y, T_2y) \neq 0$ where $p \geq 1$, $b \geq 0$ and $1 < c < 2$ or $d(T_1x, T_2y) = 0$ otherwise, then T_1 and T_2 have a unique common fixed point u .

Proof. The existence follows from the theorem 2. Now suppose that T_1 and T_2 have a second fixed point u' . Then $d(u, T_1u) + d(u', T_2u') = 0$ implies $d(T_1u, T_2u') = 0$ and so $u = T_1u$, $u' = T_2u'$ and $T_1u = T_2u'$ and so the common fixed point of T_1 and T_2 is in this case unique.

We note that without the extracondition " $d(x, T_1x) + d(y, T_2y) = 0$ implies $d(T_1x, T_2y) = 0$ " the common fixed point is not necessarily unique. (Ex., pp. 40, [1].)

Remark. If $p=1$ then theorem 1 is obtained.

Theorem 5. Let (X, d) be a complete metric space and $\{T_n\}_{n \in \mathbb{N}}$ a sequence of multifunctions of X into $CB(X)$ such that

$$(4) \quad H^p(T_1x, T_ny) \leq \frac{cd(x, T_1x).d^p(y, T_ny) + bd(x, T_ny).d^p(y, T_1x)}{\delta(x, T_1x) + \delta(y, T_ny)}$$

holds for x, y in X for which $\delta(x, T_1x) + \delta(y, T_ny) \neq 0$, where $n \geq 2$, $p \geq 1$, $b \geq 0$, $1 < c < 2$, then $\{T_n\}_{n \in \mathbb{N}}$ has a common fixed

point and $F(T_1) = F(T_n)$.

The proof follows by theorem 2 and lemma.

Let X be a nonempty set and e and d two metrics on X and $f: X \rightarrow X$ a single valued mapping. For such mappings Maia [5] proved a fixed point theorem which was generalized in many directions by Iséki [6], I.A. Rus [7],[8], K.L. Singh [9] and others. I gave in [10] and [11] some generalizations of Maia's theorem for multifunctions.

Now we prove a fixed point theorem for a sequence of multifunctions in a set with two metrics.

Theorem 6. Let X be a metric space with two metrics e and d . If X satisfies the following conditions:

- (1) $e(x,y) \leq d(x,y)$; $\forall x,y \in X$,
- (2) X is complete with respect to e ,
- (3) two multifunctions $T_1, T_2: X \rightarrow X$ are punctually closed and punctually bounded with respect to both metrics,
- (4) T_1 or T_2 is u.s.e. with respect to e ,
- (5) the inequality (2) holds for all x, y in X for which $\sigma^p(x, T_1 x) + \sigma^q(y, T_2 y) > 0$, where $p \geq 1$, $q \geq 0$, $1 < e < 2$, then T_1 and T_2 have a common fixed point and $F(T_1) = F(T_2)$.

Proof. Analogously as in the proof of the theorem 2, for any $x_0 \in X$ we can construct a sequence $\{x_n\}$ such that $x_{2n+1} \in T_1 x_{2n}$, $\{x_n\}$ being a Cauchy sequence with respect to d . Therefore, by $e \leq d$, $\{x_n\}$ is a Cauchy sequence with respect to e and since X is complete with respect to e , $x_n \rightarrow x$. As T_1 is u.s.c. from the theorem 4 [10], T_1 has a closed graph and then from $x_{2n+1} \in T_1 x_{2n}$ it results $x \in T_1 x$ and from lemma $F(T_1) = F(T_2)$.

Theorem 7. Let X be a metric space with two metrics e and d . If X satisfies the following conditions:

(1) The sequence of multifunctions $\{T_n\}_{n \in \mathbb{N}}$ is formed by punctually closed and punctually bounded multifunctions with respect to both metrics,

(2) e , d and T_1 satisfy conditions (1),(2) and (4) of theorem 6,

(3) the inequality (4) holds for all x, y in X for which $d^c(x, T_1 x) + d^c(y, T_n y) \neq 0$, where $n \geq 2$, $p \geq 1$, $b \geq 0$, $1 < e < 2$, then $\{T_n\}_{n \in \mathbb{N}}$ has common fixed points and $F(T_1) = F(T_n)$.

The proof follows by theorem 6 and lemma.

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(Oblatum 17.10. 1983)