Commentationes Mathematicae Universitatis Carolinae

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Commentationes Mathematicae Universitatis Carolinae, Vol. 24 (1983), No. 4, 681--691

Persistent URL: http://dml.cz/dmlcz/106266

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24,4 (1983)

POSITIVE QUASI-MINIMA

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Abstract: The nonnegative quasi-minima of $\int |Du|^m dx$ on a region $\Omega \subset R^n$ are investigated. The m-capacity of zero sets (excluding the trivial case) vanishes. The strong minimum principle holds uf $m \geq n-1$, the Harnack inequality and the strong Liouville theorem are valid for $m \geq n$.

Key words: Quasi-minimum, minimum principle, Harnack inequality, capacity.

Classification: 35J20, 35B50

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1. Introduction. The standard way of the variational calculus passes through the Euler equations of functionals. M. Giaquinta and E. Giusti ([3],[4]) have shown a direct method how to investigate the qualitative behavior of minima of the functional

$$F(u) = \int_\Omega f(x,u(x),Du(x))dx$$

where $\Omega \subset R^n$ is a bounded domain and f is a "reasonable" Carathéodory function, u runs through $H^{1,m}_{loc}(\Omega)$. Their method extends to the context of quasi-minima. This concept has been introduced by M. Giaquinta and E. Giusti in [3] and further studied by the same authors in [4]. It includes among others solutions of elliptic equations in divergence form (even nonvariational, without restrictions concerning the continuity of coefficients).

The main objective of [3] and [4] consists in regularity results. Nevertheless, quasi-minima possess a number of further interesting properties of the kind which is typical for solutions of elliptic equations. Several of them (e.g. removable singularities, the weak minimum principle, the Liouville theorem) are discussed in [4]. The validity of the Harnack inequality and of the strong minimum principle is stated in [4] to be open.

The present paper is devoted to a partial solution of the above mentioned problems. We consider (scalar) quasi-minima of the functional $\int |Du|^m dx$ where $m>1$. We prove that the strong minimum principle holds provided $m \geq n-1$ and the Harnack inequality is valid for $m \geq n$. By the way, the m-capacity of zero sets of nonnegative quasi-minima is investigated. Using the Harnack inequality, the Liouville theorem from [4] is strengthened.

2. Preliminaries. We start with some less obvious terminology and notation. By a continuum we understand a compact connected subset of R^n with at least two points. The k-dimensional Lebesgue or surface measure is denoted by μ_k. If u is a function, we define $Z(u) = \{x:u(x) = 0\}$. Sometimes we express points of R^n in the form $x = (x^{(1)}, x^{(2)})$ where $x^{(1)} \in R$, $x^{(2)} \in R^{n-1}$. If $A \subset R^n$, then $A^{(j)}$ means $\{x^{(j)}; x \in A\}$ $(j = 1,2)$.

Let $\Omega \subset R^n$ be a bounded set and $K \subset \Omega$ be compact. Denote

$$\text{m-cap}\,(K,\Omega) = \inf\,\{\int_\Omega |Du|^m dx ; u \in \mathcal{D}(\Omega),\ u = 1 \text{ on } K\}.$$

The set function m-cap: $K \mapsto$ m-cap(K,Ω) is termed a capacity.

2.1. Remarks. a) The domain of the capacity can be extended to more general sets than compacts, see [1]. We shall not need this fact.

b) The Hausdorff dimension of sets of m-capacity zero is less or equal to n-m (see [10]). In particular, if $m > n-1$, then every continuum in $\Omega \subset R^n$ has a positive m-capacity.

Let $u: \Omega \longrightarrow R$ be a function and $Q \geq 1$. We say that u is a Q-minimum of $\int |Du|^m dx$ on Ω if $u \in H^{1,m}_{loc}(\Omega)$ and for every open set $U \subset\subset \Omega$ and function $v \in H^{1,m}_{loc}(\Omega)$ with $v = u$ on $\Omega \setminus U$ the inequality

$$\int_U |Du|^m dx \leq Q \int_U |Dv|^m dx$$

holds. If the constant Q is not specified, we say "quasi-minimum". The collection of all nonnegative Q-minima (quasi-minima) of $\int |Du|^m dx$ on Ω is denoted by $Q^m_+(\Omega, Q)$ $(Q^m_+(\Omega))$.

We mention some properties of quasi-minima proved in [4].

2.2. Proposition. Let us be a quasi-minimum of $\int |Du|^m dx$ on Ω and $\Omega' \subset\subset \Omega$ be an open set. Then there is $v \in C^{0,\alpha}(\Omega')$ $(\alpha \in]0,1[)$ such that $u = v$ a.e.

2.3. Remark. In what follows, by quasi-minima we understand their continuous "representatives", which is justified by 2.2.

2.4. Proposition (weak minimum principle). Let u be a quasi-minimum of $\int |Du|^m dx$ on Ω and $\Omega' \subset\subset \Omega$ be an open set. Then

$$\inf_{\Omega'} u = \inf_{\partial\Omega'} u, \quad \sup_{\Omega'} u = \sup_{\partial\Omega'} u.$$

3. Smallness of zero sets. In this section $\Omega \subset R^n$ will be a bounded region.

3.1. Lemma. Let $0 \neq u \in Q^m_+(\Omega, Q)$. Then $\mu_n Z(u) = 0$.

Proof: Let $F \subset \Omega$ be the smallest closed set such that $u > 0$ a.e. on $\Omega \setminus F$. If $F = \emptyset$ we are through. Suppose $z \in \partial F \cap \Omega$. Let $B_0 \subset\subset \Omega$ be a ball with center at z. There are

$r,h>0$ and disjoint balls $B_1,B_2\subset B_0$ with radius r such that

$$\mu_n(B_1\cap Z(u))>0$$

and $u>h$ on B_2. Choose a co-ordinate system such that $(0,0)$ is a center of B_1 and $(1,0)$ is a center of B_2. Denote $M = Z(u)\cap B_1$. For every $s\in M^{(2)}$ and $k\in N$, $k>1/h$ put

$$\varphi(s) = \sup\{t\in]-r,1[:(t,s)\in B_0\cap Z(u)\},$$
$$\psi_k(s) = \inf\{t\in]\varphi(s),1[:u(t,s)\geq 1/k\}.$$

Obviously, φ, ψ_k are measurable, $u\circ\varphi = 0$, $u\circ\psi_k = 1/k$ and $\psi_k \longrightarrow \varphi$. For every $s\in M^{(2)}$ we have

$$1\leq \left(\int_{\varphi(s)}^{\psi_k(s)} k|Du(t,s)|dt\right)^m \leq$$
$$\leq (\psi_k(s)-\varphi(s))^{m-1}k^m \int_{\varphi(s)}^{\psi_k(s)} |Du(t,s)|^m dt.$$

Let $v\in\mathcal{D}(\Omega)$ be nonnegative, $v = 1$ on B_0. Denote

$$U_k = \{x\in\Omega : ku(x)<v(x)\}.$$

By the quasi-minimum property of ku we have

$$\int_{M^{(2)}}|\psi_k-\varphi|^{1-m}ds \leq \int_{U_k} k^m|Du|^m dx \leq Q\int_{U_k}|Dv|^m dx \leq$$
$$\leq Q\int_{\Omega}|Dv|^m dx.$$

However, from the monotone convergence theorem we deduce

$$\int_{M^{(2)}}|\psi_k-\varphi|^{1-m}ds \longrightarrow +\infty$$

which is a contradiction.

3.2. Theorem. Let $0\not\equiv u\in Q_+^m(\Omega,Q)$. Let $K\subset Z(u)$ be a compact set. Then $m\text{-cap}(K,\Omega) = 0$.

Proof: Find $v\in\mathcal{D}(\Omega)$, $v = 1$ on K. Let $k\in N$. Denote

$$v_k = v - \min(ku,v),$$
$$U_k = \{x\in\Omega : ku(x)<v(x)\}.$$

Although v_k need not belong to $\mathfrak{D}(\Omega)$, by a usual regularization technique we obtain

$$\text{m-cap}(K,\Omega) \leq \int_{\Omega} |Dv_k|^m dx.$$

From the quasi-minimum property of ku it follows

$$(\text{m-cap}(K,\Omega))^{1/m} \leq \left(\int_{\Omega} |Dv_k|^m dx\right)^{1/m} =$$

$$= \left(\int_{U_k} |Dv - Dku|^m dx\right)^{1/m} \leq \left(\int_{U_k} |Dv|^m dx\right)^{1/m} + \left(\int_{U_k} |Dku|^m dx\right)^{1/m} \leq$$

$$\leq (1 + Q^{1/m})\left(\int_{U_k} |Dv|^m dx\right)^{1/m}.$$

By 3.1

$$\mu_n\left(\bigcap_k U_k\right) = 0$$

and hence

$$\text{m-cap}(K,\Omega) \leq c \inf_k \int_{U_k} |Dv|^m dx = 0.$$

3.3. Remark. Treating a more general concept of capacity (cf. 2.1.a) we can simply say "m-capacity of Z(u) is zero". Using 2.1.b we conclude that the n-p-dimensional Hausdorff measure of Z(u) is zero for each $p < m$.

3.4. Lemma. Let $u \in Q_+^m(\Omega)$, $Z(u) \neq \emptyset$. Then Z(u) contains a continuum.

Proof: Choose $z \in Z(u)$. Let B be an open ball with $z \in B \subset\subset \Omega$. Denote by K the component of the set $Z(u) \cap \overline{B}$ containing z. Assume $K = \{z\}$. Then (cf. [2], Theorem 6.1.23) there is an open set U such that $z \in U \subset B$ and $u > 0$ on ∂U, which contradicts the weak minimum principle (2.4).

3.5. Corollary. Let $0 \not\equiv u \in Q_+^m(\Omega)$, $m > n-1$. Then $Z(u) = \emptyset$.

Proof: It follows from 2.1.b, 3.2 and 3.4.

4. The strong minimum principle. In this section $\Omega \subset R^n$ will be a region.

The classical strong minimum principle says that every nonnegative harmonic function on Ω vanishes identically provided it vanishes at some point. This result has been extended to more general elliptic equations of the second order by E. Hopf [6]. For further comments we refer to [5].

We present a strong minimum principle for quasi-minima of $\int |Du|^m dx$ provided $m \geq n-1$. For $m > n-1$ see 3.5. This section is devoted to $m = n-1$. The case $m < n-1$ remains open.

4.1. <u>Lemma</u>. Let $K \subset \Omega$ be a compact set with $\mu_1 K^{(1)} > 0$. Then there is a closed set $F \subset K^{(1)}$ and a continuous mapping $f: F \to R^{n-1}$ such that graph $f \subset K$ and $\mu_1 F > 0$.

<u>Proof</u>: Choose $\varepsilon \in]0, \mu_1 K^{(1)}[$. We shall construct a sequence $\{K_k\}$ of compact sets by induction. Put $K_1 = K$. If K_k is defined, we see there are compact sets $H_{k,j} \subset K_k$, $j = 1,\dots,p_k$, such that diam $H_{k,j} \leq 2^{-k}$, the sets $H_{k,j}^{(1)}$ $(j = 1,\dots,p_k)$ are pairwise disjoint and

$$\mu_1 (K_k^{(1)} \setminus \bigcup_j H_{k,j}^{(1)}) < 2^{-k}\varepsilon .$$

Put

$$K_{k+1} = \bigcup_j H_{k,j}.$$

The intersection $\bigcap_k K_k$ is the graph of a mapping with the desired properties.

4.2. <u>Theorem</u>. Let $0 \not\equiv u \in Q_+^m(\Omega, Q)$, $m = n-1$. Then $Z(u) = \emptyset$.

<u>Proof</u>: Suppose $Z(u) \neq \emptyset$. By 3.4 there is a continuum $K \subset Z(u)$. Making a suitable choice of the co-ordinate system we may suppose $\mu_1 K^{(1)} > 0$. According to 4.1 there is a closed set $F \subset K^{(1)}$ and a continuous mapping $f: F \to R^{n-1}$ such that $\mu_1 F > 0$ and graph $f \subset K \subset Z(u)$. Put

$$R = \frac{1}{2} \operatorname{dist} (\partial\Omega, \text{graph } f)$$

and

$$\Omega' = \{x \in \Omega : \mathrm{dist}(x, \mathrm{graph}\ f) < R\}.$$

Find $v \in \mathcal{D}(\Omega)$ with $v = 2$ on Ω'. Denote $S = \{x \in R^{n-1} : |x| = 1\}$. Define a mapping $\psi : F \times S \times [0,R] \to \Omega'$ by

$$\psi(t,s,r) = (t, f(t) + rs).$$

Denote

$$u_k = \min(2^k u, v),$$
$$U_k = \{x \in \Omega : u_k(x) < v(x)\},$$
$$A_k = \psi^{-1}(U_k),\ P_k = A_k \setminus A_{k+1}$$

$(k \geq 1)$. There is $j \in N$ and a set $M \subset F \times S$ such that $\mu_{n-1} M > 0$ and for every $(t,s) \in M$ and $k \geq j$ the inclusion

$$[1,2] \subset u_k \circ \psi \quad (\{t\} \times \{s\} \times [0,R])$$

holds. Then for $(t,s) \in M$ and $k > j$ we have

$$1 \leq \int_{(\{t\} \times \{s\} \times [0,R]) \cap P_k} |Du_k \circ \psi|\ dx.$$

We shall use subsequently the Hölder inequality, the change of variables for fixed t and finally the quasi-minimum property of $2^k u$ to obtain

$$(\mu_{n-1} M)^{n-1} \leq \left(\int_{P_k} |Du_k \circ \psi|\, dt\,ds\,dr\right)^{n-1} \leq$$
$$\leq \int_{P_k} r^{n-2} |Du_k \circ \psi|^{n-1} dt\,ds\,dr \left(\int_{P_k} r^{-1} dt\,ds\,dr\right)^{n-2} \leq$$
$$\leq \left(\int_{P_k} r^{-1} dt\,ds\,dr\right)^{n-2} \int_{U_k} |Du_k|^{n-1} dx \leq$$
$$\leq Q\left(\int_{P_k} r^{-1} dt\,ds\,dr\right)^{n-2} \int_{U_k} |Dv|^{n-1} dx.$$

Since by 3.1

$$\mu_n\left(\bigcap_k U_k\right) \leq \mu_n Z(u) = 0,$$

we have

$$\lim_k \int_{U_k} |Dv|^{n-1} dx = 0$$

and thus

$$\lim_{k} \int_{P_k} r^{-1} dtdsdr = +\infty .$$

Consequently, the Cesar sums

$$k^{-1} \int_{A_j \setminus A_k} r^{-1} dtdsdr$$

also tend to infinity. By 2.2 there are constants $c_1 > 0$, $\alpha \in]0,1[$ such that for every $(t,s,r) \in P_k$ we have

$$|u(\psi(t,r,s)) - u(\psi(t,s,0))| \leq c_1 r^{\alpha}$$

which implies

$$r \geq (2^{-k-1} c_1^{-1})^{1/\alpha}.$$

Hence for $k > j$ we obtain

$$\int_{A_j \setminus A_k} r^{-1} dtdsdr \leq c_2 \int_{(2^{-k-1} c_1^{-1})^{1/\alpha}}^{R} r^{-1} dr \leq c_3 k,$$

which is a contradiction.

5. <u>The Harnack inequality</u>. Let Ω, $\Omega' \subset R^n$ be regions, $\Omega' \subset\subset \Omega$. Then there is a constant c such that for every non-negative harmonic function u on Ω we have

$$\sup_{\Omega'} u \leq c \inf_{\Omega'} u.$$

This property of the Laplace equation is called the Harnack inequality. It has been generalized to more general elliptic equations of the second order by J.Moser [7], J. Serrin [8] and N.S. Trudinger [9]. For further informations and comments see [5].

We shall prove the Harnack inequality for quasi-minima of $\int |Du|^m dx$ provided $m \geq n$. The case $m < n$ remains open.

In what follows, B_ρ denotes the open ball with center at the origin and diameter ρ and S_ρ its boundary (sphere).

5.1. <u>Lemma</u>. Suppose $m \geq n$, $Q \geq 1$, $R > 0$. Then there is $r > 0$ such that for every $u \in Q_+^m(B_{2R}, Q)$ we have

$$\sup_{B_r} u \leq 2 \inf_{B_r} u.$$

<u>Proof</u>: Find $v \in \mathcal{D}(B_{2R})$, $v = 1$ on B_R. Denote $I = \int_{B_{2R}} |Dv|^m dx$. We shall use the following slight modification of the Sobolev imbedding theorem: There is $c_1 > 0$ such that

$$(\operatorname{osc}_{S_1} w)^m \leq c_1 \int_{S_1} |Dw|^m d\mu_{n-1}(x)$$

for every $w \in H^{1,m}(S_1)$. By a homothety argument we obtain for every $\rho > 0$ and $w \in H^{1,m}(S_\rho)$

$$(\operatorname{osc}_{S_\rho} w)^m \leq c_1 \rho^{m-n+1} \int_{S_\rho} |Dw|^m d\mu_{n-1}(x).$$

Find $c_2, r > 0$ such that

$$2^m c_1 Q I \leq c_2^m < \int_r^R \rho^{n-m-1} d\rho .$$

Consider $u \in Q_+^m(B_{2R}, Q)$. Denote

$$a = \inf_{B_r} u, \quad b = \sup_{B_r} u,$$

$$\tilde{u} = \min(u, b),$$

$$U = \{x \in B_{2R}, u(x) < bv(x)\}.$$

By the weak minimum principle 2.4 for every $\rho \in [r, R]$ there are $y, z \in S_\rho$ with $u(y) = a$, $u(z) = b$. We have

$$(b-a)^m \leq c_1 \rho^{m-n+1} \int_{S_\rho} |D\tilde{u}|^m d\mu_{n-1}(x) =$$

$$= c_1 \rho^{m-n+1} \int_{S_\rho \cap U} |Du|^m d\mu_{n-1}(x).$$

Using the quasi-minimum property of u we obtain

$$c_2(b-a) \leq \left(\int_r^R \rho^{n-m-1} (b-a)^m d\rho \right)^{1/m} \leq$$

$$\leq \left(c_1 \int_U |Du|^m dx\right)^{1/m} \leq \left(c_1 Q \int_U |Dbv|^m dx\right)^{1/m} \leq$$

$$\leq (c_1 QI)^{1/m} b \leq \frac{1}{2} c_2 b,$$

so $b \leq 2a$.

5.2. <u>Theorem</u>. Let Ω, $\Omega' \subset R^n$ be regions, $\Omega' \subset\subset \Omega$. Assume $m \geq n$, $Q \geq 1$. Then there is a constant $c > 0$ such that

$$\sup_{\Omega'} u \leq c \inf_{\Omega'} u$$

for every $u \in Q_+^m(\Omega, Q)$.

<u>Proof</u>: It follows from 5.1 by the usual "chaining argument", see e.g. [5], Theorem 2.5.

5.3. <u>Corollary</u> (the Liouville theorem). Suppose $m \geq n$. Then every $u \in Q_+^m(R^n)$ is constant.

<u>Proof</u>: It is easy to see that the class $Q_+^m(R^n, Q)$ (Q is fixed) is invariant under a homothety. Hence the ratio r/R from 5.1 does not depend on R and every $u \in Q_+^m(R^n, Q)$ is bounded by $2u(0)$. Now we can use [4], Theorem 4.4.

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(Oblatum 4.10. 1983)