

Bogdan Rzepecki

Some fixed point theorems for multivalued mappings

Commentationes Mathematicae Universitatis Carolinae, Vol. 24 (1983), No. 4, 741--745

Persistent URL: <http://dml.cz/dmlcz/106271>

Terms of use:

© Charles University in Prague, Faculty of Mathematics and Physics, 1983

Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.



This paper has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* <http://project.dml.cz>

SOME FIXED POINT THEOREMS FOR MULTIVALUED MAPPINGS
Bogdan RZEPECKI

Abstract: Let E be a Banach space, K a nonempty closed convex subset of E . Suppose that we have a continuous operator T which maps K into a compact subset of E and an operator F from $\overline{TK} \times K$ into K . Melvin [7] proved that if for each x , $F(x, \cdot)$ is continuous and $F(\cdot, x)$ is contraction, then the equation $F(T(x), x) = x$ has a solution. The purpose of this note is to give some generalizations of the Melvin's result for multivalued mappings.

Key words: Multivalued mappings, fixed point, uniformly convex Banach space.

Classification: 54C60, 47H10

Let E be a Banach space, K a nonempty closed convex subset of E . Suppose that we have a continuous operator T which maps K into a compact subset of E and an operator F from $\overline{TK} \times K$ into K . W.R. Melvin [7] proved that if for each x , $F(x, \cdot)$ is continuous and $F(\cdot, x)$ is contraction, then the equation $F(T(x), x) = x$ has a solution. For $F(x, y) = x + G(y)$ we obtain the fixed point theorem of Krasnoselskii [6] which combines both the Banach contraction principle and the Schauder fixed point theorem.

Problems of the said paper are in a tight connection with the results of J. Daneš (especially Proposition 9, Theorem 9 and the consequent Remarks 1 - 6, pp. 34-37) in the work [2]. These results should be made generalized for multivalued mappings. In this note, we consider the relation $x \in F(T(x), x)$ with F taking

values in the family of nonempty bounded closed convex subsets of an uniformly convex Banach space. For other generalizations see [3] - [5], [8] and [9].

The set of all nonempty subsets of a set X is denoted by 2^X . Let $\mathfrak{K}(X)$ be the family of all nonempty closed bounded convex subsets of a real normed linear space X . $\mathfrak{K}(X)$ will be regarded as a metric space endowed with the Hausdorff metric d_H , i.e.

$$d_H(A,B) = \max \left(\sup_{x \in A} d(x,B), \sup_{x \in B} d(x,A) \right);$$

here the distance between any point $x \in X$ and subset Z of X is denoted by $d(x,Z)$ ($= \inf \{ \|x - z\| : z \in Z \}$). For metric spaces X and Y , $C(X,Y)$ stands for the space of continuous bounded functions from X to Y endowed with the usual supremum metric σ .

We begin with

Definition. Let X and Y be metric spaces and let Φ be a real-valued nonnegative function defined on $C(X,Y)$. A multi-valued mapping $F: X \times Y \rightarrow 2^Y$ is called a K_Φ -mapping if (1) for each fixed $x \in X$, $F(x, \cdot)$ is closed on Y (i.e. $y_n \rightarrow y_0$ and $z_n \rightarrow z_0$ with $z_n \in F(x, y_n)$ for $n \geq 1$ implies that $z_0 \in F(x, y_0)$), and (2) for every $f \in C(X,Y)$ there exists $h_f \in C(X,Y)$ such that $h_f(x) \in F(x, f(x))$ for $x \in X$ and $\sigma(f, h_f) \leq \Phi(f) - \Phi(h_f)$.

Proposition. Let X be a metric space, Y a complete metric space and let $F: X \times Y \rightarrow 2^Y$ be a K_Φ -mapping. Then there is a function $h \in C(X,Y)$ with $h(x) \in F(x, h(x))$ for $x \in X$.

Proof. Let $h_0 \in C(X,Y)$. Since F is a K_Φ -mapping we obtain $h_n \in C(X,Y)$ ($n = 1, 2, \dots$) such that $h_n(x) \in F(x, h_{n-1}(x))$ for $x \in X$, and

$$\sigma(h_k, h_m) \leq \sum_{i=k}^{m-1} \sigma(h_i, h_{i+1}) \leq \Phi(h_k) - \Phi(h_m)$$

for $0 \leq k \leq n$. Hence $\Phi(h_0) \geq \Phi(h_1) \geq \Phi(h_2) \geq \dots$, and consequently (h_n) is a Cauchy sequence in $C(X, Y)$.

Let $h = \lim_{n \rightarrow \infty} h_n$ in $C(X, Y)$. For $x \in X$, we have $h_n(x) \rightarrow h(x)$ as $n \rightarrow \infty$ and $h_{n+1}(x) \in F(x, h_n(x))$ ($n = 1, 2, \dots$) and, since $F(x, \cdot)$ is closed, it follows that $h(x) \in F(x, h(x))$.

Now we are able to state the following

Theorem 1. Suppose we are given: E - a Banach space, Y - a nonempty closed convex subset of E , and T - a singlevalued continuous mapping from Y into a compact subset of E . If $F: \overline{T[Y]} \times Y \rightarrow 2^Y$ is a K_Φ -mapping, then the relation $x \in F(T(x), x)$ has a solution in Y .

Proof. Put $X = \overline{T[Y]}$. By Proposition there exists $h \in C(X, Y)$ with $h(x) \in F(x, h(x))$ for each $x \in X$. Now, we consider the continuous mapping $x \mapsto h(T(x))$ of Y into itself. It can be easily seen that this operator has values in a compact subset of E . Applying Schauder's theorem we infer that there is a point x_0 in Y such that $h(T(x_0)) = x_0$. Consequently $x_0 \in F(T(x_0), h(T(x_0))) = F(T(x_0), x_0)$, which completes the proof.

The Lemma below is due to Bank and Jacobs [1] and is basic in the proof of the next result.

Lemma. Let E be a uniformly convex Banach space and X a metric space, If $G: X \rightarrow \mathcal{F}(E)$ is continuous, then there is a unique continuous function $g: X \rightarrow E$ such that $g(x) \in G(x)$ and $\|g(x)\| = \inf \{\|y\| : y \in G(x)\}$ for each x in X .

Theorem 2. Let E be a uniformly convex Banach space, K a nonempty closed and convex subset of E , T a singlevalued continuous mapping from K into a compact subset of E . Suppose that F is a mapping from $\overline{T[K]} \times K$ to $\mathcal{F}(K)$ satisfying the fol-

following conditions: (1) $F(\cdot, x)$ is continuous on $\overline{T[K]}$ for every x in K , and (2) $d_H(F(x, x_1), F(x, x_2)) \leq k \|x_1 - x_2\|$ for all x in $\overline{T[K]}$ and x_1, x_2 in K and with a constant $k < 1$.

Under our assumptions there exists a point x_0 in K such that $x_0 \in F(T(x_0), x_0)$.

Proof. Let $X = \overline{T[K]}$. First of all we note that if $f \in C(X, K)$, then $x \mapsto F(x, f(x))$ is continuous on X , and, by the Lemma, there is a unique function $h_f \in C(X, K)$ with $h_f(x) \in F(x, f(x))$ and $\|f(x) - h_f(x)\| = d(f(x), F(x, f(x)))$ on X .

Assume that $f_0 \in C(X, K)$ is a given function. By the facts above, there exists a uniquely determined sequence (f_n) of functions $f_n \in C(X, K)$ ($n = 1, 2, \dots$) such that

$$f_n(x) \in F(x, f_{n-1}(x)) \text{ and}$$

$$\|f_n(x) - f_{n-1}(x)\| = d(f_{n-1}(x), F(x, f_{n-1}(x)))$$

for all $x \in X$. Hence we obtain

$$\begin{aligned} \|f_n(x) - f_{n-1}(x)\| &\leq d_H(F(x, f_{n-2}(x)), F(x, f_{n-1}(x))) \leq \\ &\leq k \|f_{n-2}(x) - f_{n-1}(x)\| \leq \dots \leq k^{n-1} \|f_0(x) - f_1(x)\| \end{aligned}$$

and

$$\sum_{n=1}^{\infty} \mathcal{G}(f_n, f_{n-1}) \leq \mathcal{G}(f_0, f_1) \cdot \sum_{n=1}^{\infty} k^{n-1} < \infty.$$

Consequently, F is a K_{Φ} -mapping on X with the function $\Phi: C(X, K) \rightarrow [0, \infty)$ defined by setting

$$\Phi(f_0) = \sum_{n=1}^{\infty} \mathcal{G}(f_n, f_{n-1}).$$

So, our result follows from the Theorem 1.

R e f e r e n c e s

- [1] H.T. BANKS and M.Q. JACOBS: Differential calculus for multifunctions, J. Math. Anal. Appl. 29(1970), 246-272.

- [2] J. DANEŠ: On densifying and related mappings and their application in nonlinear functional analysis, Theory of nonlinear operators, Akademie-Verlag, Berlin 1974, 15-46.
- [3] O. HADŽIĆ: Some theorems on the fixed points for multivalued mappings in locally convex spaces, Bull. Acad. Polon. Sci., Sér. Sci. Math. 27(1979), 277-285.
- [4] O. HADŽIĆ: A fixed point theorem for the sum of two mappings, Proc. Amer. Soc. 85(1982), 37-41.
- [5] M. KISIELEWICZ and T. JANIK: Existence theorem for functional differential contingent equations, Ann. Polon. Math. 35(1977), 157-166.
- [6] M.A. KRASNOSELSKII: Two remarks on the method of successive approximations, Uspechi Mat. Nauk 10(1955), 123-127. (Russian).
- [7] W.R. MELVIN: Some extensions of the Krasnoselskii fixed point theorem, J. Diff. Equations 11(1972), 335-348.
- [8] B. RZEPECKI: An extension of Krasnoselskii's fixed point theorem, Bull. Acad. Polon. Sci., Sér. Sci. Math. 27(1979), 481-488.
- [9] B. RZEPECKI: A note on some fixed point theorems, Glasnik Mat. 17(1982), 304-310.

Institute of Mathematics A. Mickiewicz University, Matejki 48/
49, 60-769 Poznan, Poland

(Oblatum 20.4. 1983)