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Pointless uniformities. II: (Dia)metrization

*Commentationes Mathematicae Universitatis Carolinae*, Vol. 25 (1984), No. 1, 105--120

Persistent URL: [http://dml.cz/dmlcz/106282](http://dml.cz/dmlcz/106282)

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Abstract: Metrization theorems for pointless uniformities and weak uniformities are proved.

Key words: Uniformity on a locale, diameters, metrization.

Classification: 54B15, 54E35, 06D10

This paper is a lose continuation of the paper [6]. There we have proved the equivalence of complete regularity and uniformizability in locales and indicated a role of diameters. A system $\mathcal{D}$ of diameters gives rise to a uniformity (or, to a weak uniformity, according to how strong conditions are imposed on the diameters) $\mathcal{U}(\mathcal{D})$. We have seen, in particular, that if a locale is uniformizable at all, it is uniformizable by a $\mathcal{U}(\mathcal{D})$. The main aim of this paper is to prove metrization theorems for pointless uniformities, i.e. to show that, in fact, each uniformity on a locale is a $\mathcal{U}(\mathcal{D})$, and that it is induced by a single diameter function whenever it has a countable basis. This goal is achieved by modifying the standard metrization argument (see, e.g., [5]) and, perhaps, yields also a better insight into what is going on there.

The first, and larger, part of the article (Sections 1-3) is devoted to a discussion of various conditions one can impose on diameter functions. Section 1 contains the basic defi-
nitions and relations between the conditions. In Section 2 it is shown that in the spatial case, the metric diameters are in a natural one-one correspondence with the pseudometrics. Section 3 deals with constructions allowing to obtain stronger properties of diameters. In the last, fourth, section the induced uniformities are discussed and metrization theorems are proved.

The terminology follows the standard usage (as, e.g., in [4],[1]), in special definitions the notation and convention of [6] are preserved (with the exception of the condition (M) which now contains automatically the condition (A)).

1. Diameters

1.1. We say that a subset of a locale L is connected if
\[ \forall a,b \in S \ \exists a_1, \ldots, a_n \in S \ \text{such that} \ a_1 = a, \ a_n = b \ \text{and} \ a_i \land a_{i+1} \neq 0 \ \text{for} \ i = 1, \ldots, n-1. \]
We say that it is strongly connected if
\[ a, b \in S \Rightarrow a \lor b \neq 0. \]
The system of all connected subsets of L will be denoted by \( \text{conn}(L) \),
that of the strongly connected ones by \( \mathcal{J}(L) \).

1.2. A pre-diameter on a locale L is a function
\[ d: L \rightarrow \mathbb{R}_+ \]
(\( \mathbb{R}_+ \) is the set of the non-negative reals) such that
(i) \( d(0) = 0 \),
(ii) \( a \lor b \Rightarrow d(a) \leq d(b) \),
(iii) \( \forall \varepsilon > 0, \{ \text{a} | d(a) < \varepsilon \} \) is a cover of L.
It is said to be continuous if, moreover,
(C) for each monotone (\( \subseteq \) linearly ordered by \( \leq \)) \( S \subseteq L \),
1.3. A pre-diameter \( d \) is said to be
- a **weak diameter** if
  \[ (W): \quad \text{for } a, b \text{ such that } a \land b \neq 0, \]
  \[ d(a \lor b) \leq 2 \max (d(a), d(b)); \]
- an **additive diameter** if
  \[ (A): \quad \text{for } a, b \text{ such that } a \land b \neq 0, \]
  \[ d(a \lor b) \leq d(a) + d(b); \]
- a **star diameter** if
  \[ (\ast): \quad \text{for } S \in \mathcal{S}(L) \]
  \[ d(\lor S) \leq 2 \sup \{d(a) \mid a \in S\}; \]
- a **star-additive diameter** if
  \[ (\ast A): \quad \text{for } S \in \mathcal{S}(L), \]
  \[ d(\lor S) \leq \sup \{d(a) + d(b) \mid a, b \in S, a \neq b\}; \]
- a **strong diameter** if
  \[ (S): \quad \text{for } S \subseteq \text{conn } (L), \]
  \[ d(\lor S) \leq \sup \inf \left\{ \sum_{i=1}^{n} d(a_{i}) \mid a_{i} \in S, a_{1} = a, a_{n} = b, a_{i} \lor a_{i+1} \neq 0 \mid a, b \in S, a \neq b\right\}; \]
- a **metric diameter** if
  \[ (M): \quad (A) \text{ and} \]
  \[ \forall x \in L \forall \varepsilon > 0 \exists u, v, u \land x \neq 0 \lor v \land x \& d(u), d(v) < \varepsilon \& d(u \land v)< d(x) - \varepsilon. \]

1.4. **Remark:** The following implications are obvious

\[ S \Rightarrow (\ast A) \Rightarrow (\ast) \]

\[ (\ast) \Rightarrow (M). \]

In [6, Lemma 5.1] we have seen that \((M) \Rightarrow (\ast)\). In fact, as we will shortly see, \((M)\) is the strongest of all the mentioned requirements (and, moreover, implies continuity). In the
next section we will show that the metric diameters correspond in the spatial case exactly to the pseudometrics. Thus, they can be understood as a natural modification of the notion of distance for the purposes of general locales.

The reason why we list the other mentioned conditions on pre-diameters (and no further ones used elsewhere, e.g. in [2]) is, of course, given by the aims of the article. The condition (W) is the weakest one one needs to induce at least weak uniformities; (A) is very natural and, besides, it is a part of (W); (xA) is also very natural, probably the most intuitive of all, and it will play a technical role: a star-additive diameter can be very satisfactorily approximated by a metric one; (S) is an extension of (xA) and will appear as a consequence of (W). The condition (x) is about the minimum one needs for generating uniformities; besides, star-diameters will also play a technical role.

1.5. Theorem: A metric diameter is a continuous strong diameter.

Proof: (C): Let S ⊆ L be monotone. Take an ε > 0 and choose u, v such that d(u), d(v) < ε , u ∨ S ≠ 0 ∨ v ∨ S and d(u ∨ v) > d( ∨ S) - ε . We have x,y ∈ S such that x ≤ u ∨ 0 ∨ y ≤ v. If, say, y ≥ x, we have also u ∨ y ≠ 0. Thus,

\[ d(u ∨ v) ≤ d(y ∨ u ∨ v) ≤ d(y ∨ u) + ε ≤ d(y) + 2ε \]

so that

\[ d(y) > d( ∨ S) - 3ε . \]

Hence, sup \{d(y) | y ∈ S\} ≥ d( ∨ S). On the other hand, obviously sup d(y) ≤ d( ∨ S).

(S): Let this not hold. Then, we have an S ∈ conn (L) and an η > 0 such th
\[ d(\forall S) > \sup \{ \inf \sum_{i=1}^{\omega} d(a_i) | a_i \in S, a_1 = a, a_n = b, a_1 \wedge a_{i+1} \neq 0 \} + \eta. \]

Take an \( \varepsilon > 0 \) such that \( \varepsilon < \frac{1}{\varepsilon} \eta \) and choose \( u, v \) such that
\[ d(u), d(v) < \varepsilon, \ u \wedge \forall S \neq 0 \neq v \wedge \forall S \] and \( d(u \vee v) \geq d(\forall S) - \varepsilon. \)

Consider \( a, b \in S \) with \( u \wedge a \neq 0 \neq v \wedge b. \)

I. Let \( a \neq b. \) Then we have, in particular,
\[ d(\forall S) > \inf \sum_{i=1}^{\omega} d(a_i) | a_i \in S, a_1 = a, a_n = b, a_1 \wedge a_{i+1} \neq 0 \} + \eta. \]
and hence there are \( a_1 = a, a_2, \ldots, a_n = b, a_1 \wedge a_{i+1} \neq 0 \) such that
\[ (1) \ d(\forall S) \geq \sum d(a_i) + \frac{1}{2} \eta. \]

By 1.2(iii) we can choose \( u \in V \) such that
\[ d(u_i) < \varepsilon \] and \( u \neq a \wedge a_{i+1}. \)

We obtain
\[ d(u_1 \vee u_1) \neq d(u_1 \wedge a_1) \leq d(a_1) + \varepsilon, \]
\[ d(u_1 \vee u_2) \leq d(a_2), \]
\[ \vdots \]
\[ d(u_{n-2} \vee u_{n-1}) \leq d(a_{n-1}), \]
\[ d(u_{n-1} \vee v) \leq d(a_{n} \vee v) \leq d(a_n) + \varepsilon. \]

Using repeatedly (A) we obtain
\[ d(u \vee u_1 \vee \ldots \vee u_{n-1} \vee v) \leq d(u \vee u_1) + d(u_1 \vee u_2) + \ldots + d(u_{n-1} \vee v) \leq \sum d(a_i) + 2 \varepsilon \]
so that
\[ d(\forall S) < d(u \vee v) + \varepsilon \leq \sum d(a_i) + 3 \varepsilon < \sum d(a_i) + \frac{1}{2} \eta \]
in contradiction with (1).

II. Let \( a = b. \) Choose an arbitrary \( c \in S, c \neq a \) (obviously \( S \) has to have at least two elements). We have
\[ d(\forall S) > \inf \sum_{i=1}^{\omega} d(a_i) | a_1 = a, a_n = c, a_1 \wedge a_{i+1} \neq 0 \} + \eta \]

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so that, again, there are $a_1 = a$, $a_2, \ldots, a_n = 0$ such that
\[ d(\sqrt{S}) > \sum d(a_i) + \frac{1}{2} \eta. \]

We obtain a contradiction
\[
d(\sqrt{S}) < d(u \lor v) + \varepsilon < d(v \lor a \lor v) + \varepsilon < d(a) + 3 \varepsilon \leq \sum d(a_i) + 3 \varepsilon < \sum d(a_i) + \frac{1}{2} \eta < d(\sqrt{S}). \]

2. **Spatial case: metric diameters and pseudometrics**

2.1. In this section, a topological space $X = (X, L)$ is given, $L$ is the locale of its open sets. To keep the notation in accord with that of the general case, we will denote the open sets in $X$ by lower case Roman letters. The points of $X$ will be denoted by $\alpha$, $\beta$, $\gamma$ and $\delta$. If $\Phi$ is a pseudometric on $X$ we write
\[
\Omega^\Phi(\alpha, \beta) = \{ \beta | \Phi(\alpha, \beta) < \varepsilon \}.
\]

2.2. Let $\Phi$ be a bounded pseudometric on the set $X$. We construct
\[
d: L \to \mathbb{R}_+
\]
by putting
\[
(2) \quad d(x) = \sup \{ \Phi(\alpha, \beta) | \alpha, \beta \in x \}.
\]

2.3. **Proposition:** Let the topology of $(X, \Phi)$ be weaker than that of $X$. Then $d$ defined by (2) is a metric diameter.

**Proof** is a matter of easy checking. Since the sets $\Omega^\Phi(\alpha, \beta)$ are open, we can take for $u, v$ in $(\mathbb{M})$ suitable $\Omega(u, \frac{1}{2} \varepsilon)$, $\Omega(\beta, \frac{1}{2} \varepsilon)$. □

2.4. Let $d: L \to \mathbb{R}_+$ be a metric diameter, define
\[
\Phi: X \times X \to \mathbb{R}_+
\]
by putting
\[
(3) \quad \Phi(\alpha, \beta) = \inf \{ d(x) | \alpha, \beta \subseteq x \}.
\]
2.5. Proposition: The function $\varphi$ is a pseudometric on the set $X$.

Proof: The triangle inequality follows easily from (A), $\varphi(\alpha, \alpha) = 0$ from 1.2(iii). Obviously, $\varphi(\alpha, \beta) = \varphi(\beta, \alpha)$. □

2.6. Lemma: Let $\varphi$ be constructed from $d$ by (3). Then

$$\Omega_{\varphi}(\alpha; \varepsilon) = \vee\{x | x \in L, \alpha \in x, d(x) < \varepsilon \}.$$  

Consequently, the topology of $(X, \varphi)$ is weaker than $L$.

Proof: We have

$$\varphi(\alpha, \beta) < \varepsilon \iff \exists x \ni \{\alpha, \beta\}, x \in L, \text{ such that } d(x) < \varepsilon.$$ □

2.7. Theorem: The formulae (2) and (3) constitute a one-one correspondence between the set of all bounded metric diameters $d$ on $L$ and the set of all bounded pseudometrics $\varphi$ on $X$ such that the topology of $(X, \varphi)$ is weaker than $L$.

Proof: I. Start with a diameter $d$, construct $\varphi$ by (3) and a new diameter $d^*$ from $\varphi$ by (2). Obviously,

$$d^*(x) \leq d(x).$$

Let there be an $x$ and an $\varepsilon > 0$ such that $d(x) > d^*(x) + 3 \varepsilon$.

Take $u, v$ such that $u \land x \neq 0 \neq v \land x$, $d(u), d(v) < \varepsilon$ and $d(u \lor v) > d(x) - \varepsilon$ (and, hence, $d(u \lor v) > d^*(x) + 2 \varepsilon$). Choose $\alpha \in u \land x$, $\beta \in v \land x$. Consider an arbitrary $w \in L$ such that $\{\alpha, \beta\} \sqsubseteq w$. We have

$$d(u \lor v) \leq d(w) + 2 \varepsilon$$

and hence

$$d(w) \geq d(u \lor v) - 2 \varepsilon > d^*(x)$$

so that

$$\varphi(\alpha, \beta) \geq d(u \lor v) - 2 \varepsilon > d^*(x)$$

in contradiction with the definition of $d^*(x)$.

II. Start with a pseudometric $\varphi$, construct $d$ by (2) and then a new pseudometric $\varphi'$ by (3). We obviously have

$$\varphi'(\alpha, \beta) \geq \varphi(\alpha, \beta).$$

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Let \( \phi'(\alpha, \beta) > \phi(\alpha, \beta) + 3\varepsilon \). Consider \( u = \Omega(\alpha, \frac{1}{2} \varepsilon) \), \( v = \Omega(\beta + \frac{1}{2} \varepsilon) \). Thus, \( d(u), d(v) < \varepsilon \). Take \( \gamma \in u \), \( \delta \in v \). We have

\[
\phi(\gamma, \delta) \leq \phi(\gamma, \alpha) + \phi(\alpha, \beta) + \phi(\beta, \delta) < \phi(\alpha, \beta) + 2\varepsilon
\]

and if \( \gamma, \delta \in u \) or \( \gamma, \delta \in v \) obviously \( \phi(\gamma, \delta) < 2\varepsilon \). Thus

\[
d(u \vee v) < \phi(\alpha, \beta) + 2\varepsilon < \phi'(\alpha, \beta) - \varepsilon
\]

in contradiction with the definition of \( \phi' \). \( \square \)

2.8. Proposition: (Notation from [6].) Let \( d \) be a metric diameter on \( L \), let \( \phi \) be obtained by (3). Then

\( u \) is open in \( (X, \phi) \) iff \( u \in L_{\phi} \)

where \( L_{\phi} \) is the \( u \)-basis \( \{ \{ a \} | d(a) < \varepsilon \} \) \( \varepsilon \geq 0 \).

Proof: Let \( u \) be open in \( (X, \phi) \). By 2.6, \( u \in L_{\phi} \). Let \( \alpha \) be an arbitrary point of \( u \). Take an \( \varepsilon > 0 \) such that \( \Omega(\alpha, 2\varepsilon) \subseteq u \).

Put \( v = \Omega(\alpha; \varepsilon) \) and consider \( \Lambda = \{ a \} | d(a) < \varepsilon \} \). We have

\( \Lambda \subseteq u \) and hence \( \forall \varepsilon \leq u \).

Since \( \alpha \) was arbitrary, \( u = \sqrt{\forall \varepsilon \leq u} \).

On the other hand, let \( u = \sqrt{\forall \varepsilon \leq u} \). Take an \( \alpha \in u \).

There is an \( x, x \leq u \) such that \( \alpha \leq x \) and there has to be an \( \varepsilon > 0 \) such that, for \( \Lambda = \{ a \} | d(a) < \varepsilon \} \), \( \Lambda \subseteq x \). Obviously,

\( \Omega(\alpha; \varepsilon) \subseteq x \). \( \square \)

3. Fabricating diameters with stronger properties

3.1. For a star diameter \( d \) on a locale \( L \) put

\[
d'(x) = \inf_{\varepsilon > 0} \sup \{ d(u \vee v) | u \wedge x \neq 0 \neq v \wedge x, d(u), d(v) < \varepsilon \}
\]

3.2. Lemma: For any \( x, y \in L \) we have

\[
d'(x \vee y) \geq d(x \vee y) - d(x) - d(y)
\]

Proof: If \( x = 0 \) or \( y = 0 \), the right hand side is zero.
Thus, we can assume that \( x \neq 0 \neq y \).
Let $d(x \lor y) < d(x \lor y) - d(x) - d(y)$. Then we have an $\epsilon > 0$ such that, for $\varepsilon < \varepsilon_0$,

$$\alpha = \sup \{ d(u \lor v) | u \land (x \lor y) \neq 0 \lor (x \lor y), d(u), d(v) < \varepsilon \} < d(x \lor y) - d(x) - d(y).$$

Choose $u, v$ such that $u \land x \neq 0 \lor (x \lor y)$ and $d(u), d(v) < \varepsilon$. We have $d(x \lor y) \leq d(x \lor y \lor u \lor v) \leq d(u \lor v) + d(x) + d(y)$ and we obtain a contradiction

$$\alpha \geq d(u \lor v) \geq d(x \lor y) - d(x) - d(y) \geq \infty. \quad \square$$

3.3. **Lemma:** We have

$$\frac{1}{2}d(x) \leq \sigma(x) \leq d(x).$$

**Proof:** If $u \land x \neq 0 \lor (x \lor y)$ and $d(u), d(v) < \varepsilon$, we have

$$d(u \lor v) \leq d(x \lor u \lor v) \leq d(x) + 2\varepsilon.$$

Hence, $\sigma(x) \leq d(x)$.

Now, let us have, for some $x \in L$ and $\eta > 0$,

$$\sigma(x) < \frac{1}{2}d(x) - \eta.$$

Thus, we have an $\varepsilon > 0$ such that

$$\sup \{ d(u \lor v) | u \land x \neq 0 \lor (x \lor y), d(u), d(v) < \varepsilon \} < \frac{1}{2}(d(x) - \eta).$$

Take the system $S = \{ u \in L | d(u) \leq \varepsilon, u \land x \neq 0 \lor (x \lor y) \}$ and choose a fixed $v_0 \in S$. Thus, $x \leq \lor \{ u \lor v_0 | u \in S \}$ and we obtain, by $(\ast)$,

$$d(x) \leq 2 \sup \{ d(u \lor v_0) | u \in S \} < d(x) - \eta$$

which is a contradiction. \quad \square

3.4. **Theorem:** For any star-additive diameter $d$ there is a metric diameter $\sigma$ such that

$$\frac{1}{2}d \leq \sigma \leq d.$$

**Proof:** According to 3.3 it suffices to prove that the $\sigma$ from 3.1 is a metric diameter. Obviously, it is a prediameter (1.2(ii) is straightforward and 1.2(i) and (iii) follow from 3.3).
(A): Let it not hold. Hence, we have some \( a, b \in L \), \( a \land b \neq 0 \), and an \( \eta > 0 \) such that
\[
\sigma(a \lor b) > \sigma(a) + \sigma(b) + \eta.
\]
Thus, for a sufficiently small \( \varepsilon > 0 \),
\[
\sigma(a \lor b) > \sup \{d(u_1 \lor v_1) | u_1 \land a \neq 0 \lor v_1 \land a, \ d(u_1), \ d(v_1) < \varepsilon \} + \\
+ \sup \{d(u_2 \lor v_2) | u_2 \land b \neq 0 \lor v_2 \land b, \ d(u_2), \ d(v_2) < \varepsilon \} + \frac{1}{2} \eta.
\]
Choose \( u, v, u \land (a \lor b) \neq 0 \lor v \land (a \lor b), \ d(u), \ d(v) < \varepsilon \) such that
\[
\sigma(a \lor b) < d(u \lor v) - \frac{1}{2} \eta.
\]
so that
\[
d(u \lor v) > \sup \{d(u_1 \lor v_1) | \ldots \} + \sup \{d(u_2 \lor v_2) | \ldots \}.
\]
Thus, neither \( u \land a \neq 0 \lor v \land a \) nor \( u \land b \neq 0 \lor v \land b \) and we can assume \( u \land a \neq 0 \lor v \land b \). Choose a \( w \in L \) such that \( d(w) < \varepsilon \) and \( w \land (a \lor b) \neq 0 \lor 0 \). We obtain a contradiction
\[
d(u \lor v) > d(u \lor w) + d(w \lor v) \geq d(u \lor v \lor w).
\]
The metric property: By 3.2 we obtain
\[
(4) \quad \sigma(x) \leq \inf_{\varepsilon > 0} \{d(u \lor v) + 2 \varepsilon | u \land x \neq 0 \lor v \land x, \ d(u), \ d(v) < \varepsilon \}.
\]
Let \( \sigma \) be not metric. Then we have an \( \varepsilon_0 > 0 \) such that for all \( u, v \) such that \( \sigma(u), \ \sigma(v) < \varepsilon_0 \) and \( u \land x \neq 0 \lor v \land x \) necessarily
\[
(5) \quad \sigma(u \lor v) < \sigma(x) - \varepsilon_0.
\]
Choose an \( \varepsilon < \frac{1}{2} \varepsilon_0 \). By (4) and 3.3 we have
\[
\sigma(x) \leq \sup \{d(u \lor v) + 2 \varepsilon | \ldots, \ \sigma(u), \ \sigma(v) < \varepsilon \}
\]
and hence, using (5), we obtain a contradiction
\[
\sigma(x) < \sigma(x) - \varepsilon_0 + 2 \varepsilon < \sigma(x). \quad \square
\]
3.5. Let \( f \) be a pre-diameter. For Secoll (L) and \( a, b \in S \)
put \( \mu_f(a, b, S) = \inf \sum_{i=1}^{N} f(a_i)^{a_1} = a, a_n = b, a_1 \land a_{i+1} \neq 0, \ a_1 \land b \in S \}.
\]
Further, put
\[
\mu_f(S) = \sup \{\mu_f(a, b, S) | a, b \in S \}.
\]
3.6. **Observation**: 1. Let \( b_1 \land b_2 = 0 \). Then
\[
\mu(a, c, S) = \mu(a, b_1, S) + \mu(b_2, c, S).
\]
2. Let \( S_1 \subseteq S_2 \). Then
\[
\mu(a, b, S_1) \geq \mu(a, b, S_2).
\]
3.7. For \( x \in L \) put
\[
d_x(x) = \inf \{ \mu_x(S) | S \in \text{conn} (L), x \leq \sqrt{S} \}.
\]
Obviously,
\[
d_x \leq f.
\]

3.8. **Theorem**: The function \( d_x \) is a star-additive diameter.

**Proof**: Obviously, \( d_x \) is a pre-diameter. Let it not be star-additive. Thus, we have an \( S \in \mathcal{S}(L) \) and an \( \varepsilon > 0 \) such that
\[
d_x(\sqrt{S}) > \sup \{ d(a) + d(b) | a, b \in S, a \neq b \} + 3\varepsilon.
\]
For each \( a \in S \) choose an \( S_a \in \text{conn} (L) \) such that
\[
\sqrt{S} \supseteq a \quad \text{and} \quad \mu_x(S_a) < d_x(a) + \varepsilon.
\]
Thus, by (6), we have
\[
d(\sqrt{S}) > \mu(S_a) + \mu(S_b) + \varepsilon.
\]
Put \( T = \bigcup \{ S_a | a \in S \} \). Obviously, \( T \in \text{conn} (L) \) and \( \sqrt{T} \supseteq \sqrt{S} \) so that
\[
\mu(T) \geq d(\sqrt{S}) \quad \text{and hence, by (7),}
\]
\[
\mu(T) > \mu(S_a) + \mu(S_b) + \varepsilon.
\]
Thus, there exist \( u, v \in T \) such that
\[
\mu(u, v, T) > \mu(S_a) + \mu(S_b).
\]
We cannot have \( u, v \in S_a \) for an \( a \), since then we would have (see 3.6.2) \( \mu(u, v, T) \leq \mu(u, v, S_a) \leq \mu(S_a) \). Thus, there are \( a, b \), \( a \neq b \), \( u \in S_a \) and \( v \in S_b \). Choose an \( x \in S_a \) and a \( y \in S_b \) such that \( x \land y = 0 \). Now, (8) and 3.6 yield a contradiction
\[
\mu(u, v, T) > \mu(u, x, S_a) + \mu(y, v, S_b) \geq \mu(u, x, T) + \mu(y, v, T) \geq \mu(u, v, T).
\]

3.9. We will formulate one more condition concerning pre-diameters \( f \):
Lemma: Let $f$ satisfy (3W). Let $x_1, \ldots, x_n$ be such that $x_i \neq x_{i+1}$ for $i = 1, \ldots, n-1$. Then

$$f(\sqrt[n]{x_1}) \leq 2 \sum_{i=1}^{n} f(x_i).$$

Proof by induction on $n$. For $n = 1$, $f(x_1) \leq 2f(x_1)$. Let the inequality hold for $n$, consider $x_1, \ldots, x_{n+1}$. Put $\alpha = \sum_{i=1}^{n} f(x_i)$ and take the first $k$ such that $\sum_{i=1}^{k} f(x_i) \geq \frac{1}{2}\alpha$. Then

$$\sum_{i=1}^{k-1} f(x_i) < \frac{1}{2}\alpha, \quad \sum_{i=k+1}^{n} f(x_i) \leq \frac{1}{2}\alpha$$

and hence, by the induction hypothesis,

$$f(\sqrt[n]{x_1}) \leq \alpha, \quad f(\sqrt[n]{x_{n+1}}) \leq \alpha.$$ 

Since also $f(x_{k+1}) \leq \alpha$ we obtain, using (3W),

$$f(\sqrt[n]{x_1}) \leq 2\alpha = 2\sum_{i=1}^{n} f(x_i). \quad \Box$$

3.10. Lemma: Let $f$ be a star diameter satisfying (3W), let $\mu_f$ be the function from 3.5. Then for any $S \subseteq \text{conn}(L)$

$$f(\bigvee S) \leq 4 \mu_f(S).$$

Proof: Fix a $u \in S$ and an $\varepsilon > 0$. For each $u \in S$ choose a sequence $x_1(u), \ldots, x_n(u) \in S$ such that $u_0 = x_1(u)$, $u = x_n(u)$, $x_i(u) \neq x_{i+1}(u)$ and

$$\sum f(x_i(u)) < \mu_f(u_0, u, S) + \varepsilon.$$ 

Put $s(u) = \bigvee x_i(u)$. Evidently, $s(u) \subseteq s(v) \geq u_0 \neq 0$ and $u \neq s(u)$ so that (9)

$$\forall s(u) \uparrow u \in S \exists = \bigvee S \text{ and } \{s(u) \uparrow u \in S \} \subseteq \mathcal{I}(L).$$

By 3.9 we have

$$f(s(u)) \leq 2 \sum f(x_i(u)) < 2 \mu_f(u_0, u, S) + 2 \varepsilon \leq 2 \mu_f(S) + 2 \varepsilon$$

and hence, by (8) and (9),

$$f(\bigvee S) \leq 2 \sup \{f(s(u)) \uparrow u \in S\} \leq 4 \mu_f(S) + 4 \varepsilon. \quad \Box$$
3.11. **Theorem**  For each star diameter \( f \) satisfying (3W) there is a metric diameter \( d \) such that

\[
gf(x) \leq d(x) \leq f(x).
\]

**Proof:** Consider first the function \( d_f \) from 3.7. Let \( S \) be in \( \text{conn} (L) \), \( x \leq \vee S \). By 3.10

\[
f(x) \leq f(\vee S) \leq 4\mu_x(S)
\]

and hence

\[
d_f(x) = \inf \{ \mu_x(S) \mid \vee S \geq x \} \geq \frac{1}{4}f(x).
\]

By 3.8, \( d_f \) is star additive so that our statement now follows from 3.4. \( \square \)

### 4. (Dia)metrization of uniformities

4.1. A u-basis (resp. wu-basis) \( A \) such that \( \mathcal{K} = \mathcal{U} \) (see [6; 3.3, 3.5]) will be referred to as a basis of the uniformity (resp. weak uniformity) \( \mathcal{U} \).

It is said to be **meet-closed** if

\[
A, B \in A \Rightarrow \exists C \in A, C \leq A \land B.
\]

Obviously, if \( A \) is meet-closed then

\[
A \in \mathcal{U} \iff \exists B \in A, B \leq A.
\]

4.2. For a u-basis (wu-basis) \( A \) put

\[
mA = \{ A_1 \land \ldots \land A_k \mid A_i \in A \}.
\]

By [6; 3.4] we see that \( mA \) is a u-basis (wu-basis) again. Obviously it is meet-closed. Thus, we make an

**Observation:** If \( \mathcal{U} \) has a countable basis, it has a countable meet-closed basis. \( \square \)

4.3. **Lemma:** Let a uniformity (resp. a weak uniformity) \( \mathcal{U} \) have a countable basis. Then it has a meet closed basis \( A = \{ A_0, A_1, \ldots, A_n, \ldots \} \) such that \( A_0 = \{ \} \) and, for each \( n \),

\[
A_{n+1} \land A_n \quad \text{(resp. } A_{n+1}^{(2)} \land A_n^{(2)})
\]

**Proof:** Take a meet-closed basis \( \mathcal{B} = \{ B_1, B_2, \ldots, B_n, \ldots \} \)
of \( \mathcal{U} \). Put \( A_0 = \{ e^1 \}, A_1 = B_1 \). Let \( A_0, \ldots, A_n \) be already defined so that

(a) \( A_{k+1}^* < A_k \) (resp. \( A_{k+1}^{(2)(2)} < A_k \)) for \( k < n \),

(b) \( A_k \in \mathcal{B}_s \) for \( k \leq n \),

(c) \( A_k < B_k \) for \( k = n \).

There is a \( B_r \) such that \( B_r^* < A_n \) resp. \( B_r^{(2)(2)} < A_n \) and a

\( B_s < B_r \wedge B_{n+1} \). Put \( A_{n+1} = B_s \).

4.4. Proposition: For each uniformity (resp. weak uniformity) \( \mathcal{U} \) there is a system \( ( \mathcal{U}_i \mid i \in J ) \) of uniformities (resp. weak uniformities) with countable bases such that

\[ A \in \mathcal{U} \text{ iff } \exists i \in J \text{ s.t. } A \in \mathcal{U}_i. \]

Proof: For an \( A \in \mathcal{U} \) choose inductively \( A_1, A_2, \ldots, A_n, \ldots \)

so that \( A = A_1, A_{n+1}^* < A_n \) (resp. \( A_{n+1}^{(2)(2)} < A_n \)). Put \( J = \mathcal{U} \), \( \mathcal{A}_A = \{ A_i \mid i = 1, 2, \ldots \} \), \( \mathcal{U}_A = \mathcal{A}_A^* \).

4.5. For a weak diameter \( d \) put

\[ \mathcal{U}(d) = \{ A \mid \exists \varepsilon > 0, \{ a \mid d(a) < \varepsilon \} \in A \}. \]

More generally, let \( \mathcal{D} \) be a system of weak diameters. Put

\[ \mathcal{U}(\mathcal{D}) = \mathcal{K} \] where \( \mathcal{K} = \{ a \mid d(a) < \varepsilon \mid d \in \mathcal{D}, \varepsilon > 0 \} \)

(using \( \mathcal{A} \) has been necessary to ensure the meet property; in the case of one \( d \) this is automatic).

Obviously, \( \mathcal{U}(d), \mathcal{U}(\mathcal{D}) \) are weak uniformities. If \( d \) resp.

all the members of \( \mathcal{D} \) are star diameters, \( \mathcal{U}(d) \) resp. \( \mathcal{U}(\mathcal{D}) \) is a uniformity.

4.6. Theorem: \( \mathcal{U} \) is a uniformity with a countable basis iff

there is a metric diameter such that \( \mathcal{U} = \mathcal{U}(d) \).

(Note that this fact provides the formal definition of metrizability in [3] with a more concrete contents.)

Proof: Consider the basis \( A_0, A_1, \ldots, A_n, \ldots \) from 4.3 and define \( f: L \to \mathbb{R}_+ \) by putting

\[ f(x) = \inf \{ 2^{-n} \mid x \leq a \text{ for some } a \in A_n \}. \]
Obviously, \( f \) is a pre-diameter. Now, let \( S \) be as in \( \mathcal{G}(L) \) and let \( d(a) \leq 2^{-(n+1)} \) for all \( a \in S \). Thus, we have for each \( a \in S \) a \( b(a) \in A_{n+1}^* \) such that \( b(a) \geq a \). Hence, \( \sqrt{S} \leq \sqrt{\{b(a)\} \mid a \in S} \subseteq A_{n+1}^* \subseteq c \Lambda_{n+1}^* \setminus \Lambda_n \) so that \( \sqrt{S} \leq b \) for some \( b \in \Lambda_n \). Thus, \( f(a) \leq 2^{-(n+1)} \) for all \( a \in S \) implies \( f(\sqrt{S}) \leq 2^{-n} \) and hence \( f(\sqrt{S}) \leq 2 \sup \{ f(a) \mid a \in S \} \) so that \( f \) is a star diameter.

Now, let \( x, y, z \) be such that \( x \wedge y \neq 0 \neq y \wedge z \). If \( f(x), f(y), f(z) \leq 2^{-(n+1)} \), we have \( a, b, c \in A_{n+1}^* \) such that \( x \leq a, y \leq b, z \leq c \). Hence, \( a \wedge b \in A_{n+1}^* \) and \( a \vee b \in (A_{n+1}^* \setminus A_n) \), hence \( f(x \vee y \vee z) \leq 2^{-n} \) and we conclude that also (3W) is satisfied.

Thus, by 3.11 there is a metric diameter \( d \) such that \( 1 \leq f \leq d \leq f \).

We check easily that \( \mathcal{U} = \mathcal{U}(f) \) and that \( \mathcal{U}(f) = \mathcal{U}(d) \).

On the other hand, obviously every \( \mathcal{U}(d) \) has the countable basis \( \{ \text{ald}(a) < \frac{1}{n} \} \mid n = 1, 2, \ldots \} \). \( \square \)

4.7. **Theorem:** For every uniformity \( \mathcal{U} \) there is a set of metric diameters \( \mathcal{D} \) such that \( \mathcal{U} = \mathcal{U}(\mathcal{D}) \).

**Proof:** follows easily from 4.4 and 4.6. \( \square \)

4.8. **Remark:** The constructions of Section 3 have served the purpose of crossing the gap between the star diameters and the metric ones (of course, this has to be done if we wish to have a generalization of the well-known metrization theorems — see Section 2). To prove just that

\( \mathcal{U} \) is a uniformity with a countable basis iff there is a star diameter \( d \) such that \( \mathcal{U} = \mathcal{U}(d) \)

(and a similar weaker analogon of 4.7) one needs the first half of the proof of 4.6 only, without any reference to Section 3.

Similarly, one immediately obtains that
\[ \mathcal{U} \] is a weak uniformity with a countable basis iff there is a weak diameter \( d \) such that \( \mathcal{U} = \mathcal{U}(d) \), and that

For every weak uniformity \( \mathcal{U} \) there is a set of weak diameters \( \mathcal{D} \) such that \( \mathcal{U} = \mathcal{U}(\mathcal{D}) \).

There seems to be a problem of some interest as to whether the weak diameters in these statements can be replaced by additive ones.

References


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(Oblatum 17.1. 1984)