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Completely regular modification and products

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Abstract: If $X$ is a topological space, denote $CR(X)$ the completely regular modification of $X$. The aim of the present paper is to give an example of two $T_j$-spaces $X$, $Y$ such that $CR(X \times Y) \neq CR(X) \times CR(Y)$.

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There is a plenty of papers dealing with the commutativity of products and a suitable functor from the category of topological spaces into itself. To the author's knowledge, the functor of completely regular modification has been investigated from this point of view in [O] and [P]. For a topological space $X$, denote $CR(X)$, the completely regular modification of $X$, the space whose underlying set is the same as that of $X$, equipped with the topology, the base of which consists of all cozero subsets of $X$. It is easy to show that $CR(X)$ has the largest completely regular topology contained in the topology of $X$. Let us remind the best results concerning the commutativity of $CR$ and products:

Theorem [O]: Let $X$ be Tychonoff. Then the following are equivalent:
(i) $X$ is locally compact,
(ii) for each space $Y$, $CR(X \times Y) = X \times CR(Y)$.

**Theorem [9]:** Let $X$ be a topological space and suppose that $CR(X)$ is not locally compact. Then there exists a Hausdorff space $Y$ such that $CR(X \times Y) \neq CR(X) \times CR(Y)$.

According to these two theorems, the picture is pretty clear: local compactness is the crucial property. Unfortunately, the proof of the second theorem mentioned above essentially uses the fact that the space $Y$ is not regular.

We do not know the answer, whether "Hausdorff" can be replaced by "regular" in the second theorem of S. Oka. Nevertheless, we can exhibit the following

**Example:** There exist regular spaces $X$ and $Y$ such that $CR(X) \times CR(Y) \neq CR(X \times Y)$.

The idea is fairly simple. Let us start with a completely regular, non-normal space $T$, let $A, B \subseteq T$ be the two closed disjoint sets which cannot be separated. Run the space $T$ through the Jones machine. You will obtain the regular space $X$ which contains a point $p$ and a closed set $A_0$ isomorphic to $A$ such that $p$ and $A_0$ cannot be functionally separated. This implies that whenever $U$ is a cozero set in $X$ which contains $p$, then $U \cap A_0$ is infinite. Consequently, the point $(p, p)$ belongs to the closure of the set $\{(x, x) : x \in A_0\}$ in the space $CR(X) \times CR(X)$. In order to show that $CR(X \times X)$ differs from $CR(X) \times CR(X)$, we need to find a continuous real-valued function on $X \times X$ which vanishes in $(p, p)$ and equals 1 in each $(x, x)$, $x \in A_0$.

Unfortunately, this does not work in general and we ought to be a bit more careful when choosing the starting non-normal space — in fact, we shall need two such spaces. In spite of
A. The modified Tychonoff plank. Let $\tau \leq \omega$ be a cardinal number, let $\mathcal{F} = \{ \mathcal{F}_\alpha : \alpha \in \tau \}$ be an arbitrary family of infinite subsets of $\omega$.

The modified Tychonoff plank $T(\mathcal{F})$ is defined as follows: The underlying set is $(\tau + 1) \times (\omega + 1) - \{(\tau, \omega)\}$, every point $(\alpha, n)$ (for $\alpha < \tau$, $n < \omega$) is isolated, the neighborhood base of a point $(\alpha, n)$ (for $n < \omega$) is the collection
\[ \{i(\tau, n) \cup \{\alpha, n) : \alpha \in \tau - \{\tau\}; \mathcal{F}_\alpha \} \], \]
the neighborhood base of a point $(\alpha, \omega)$ (for $\alpha < \tau$) is the collection
\[ \{i(\alpha, \omega) \cup \{\alpha, n) : n \in \mathcal{F}_\alpha - \mathcal{F} \}; \mathcal{F} \in [\omega]^{< \omega}\}. \]
Sometimes it will be convenient to emphasize by a subscript $(\alpha, n)_\mathcal{F}$ that the pair $(\alpha, n)$ belongs to $T(\mathcal{F})$.

Now, the space $T(\mathcal{F})$ is completely regular Hausdorff $0$-dimensional. It is normal if and only if $|\mathcal{F}| \leq \omega$, because the sets $A_{\mathcal{F}} = \{\tau \times \omega$ and $B_{\mathcal{F}} = \tau \times \{\omega\}$ cannot be separated iff $\tau > \omega$.

The forthcoming lemma shows one important property of continuous functions on $T(\mathcal{F})$.

For $\mathcal{F} \subset [\omega]^{< \omega}$, denote $\mathcal{J}(\mathcal{F}) = \{X \subset [\omega]^{< \omega} : \{P \in \mathcal{F} : |P \cap X| = \omega \} \}.$

**Lemma 1.** Let $\mathcal{F} \subset [\omega]^{< \omega}$, $\tau = |\mathcal{F}| > \omega$, let $f : T(\mathcal{F}) \to \mathbb{R}$ be continuous, $\varepsilon > 0$. Then

(i) if $|i\alpha \in \tau : |f((\alpha, \omega))| \geq \varepsilon| \leq \omega$, then $\{n \in \omega : |f((\tau, n))| > \varepsilon \} \in \mathcal{J}(\mathcal{F})$, and almost conversely

(ii) if $\{n \in \omega : |f((\tau, n))| \geq \varepsilon \} \in \mathcal{J}(\mathcal{F})$, then $|i\alpha \in \tau : |f((\alpha, \omega))| > \varepsilon| \leq \omega$.

**Proof.** Since $f$ is continuous, then for each $n, k \in \omega$ the
set \( S_{n,k} = \{ \alpha \in \tau : |f((\alpha,n)) - f((\tau,n))| \leq \frac{1}{k} \} \) is countable.
Let \( S = \bigcup_{k=1}^{\infty} \bigcup_{n=0}^{\infty} S_{n,k} \), \( Z = \tau - S \). Then for \( \alpha \in Z \) and \( n \in \omega \)
\( f((\alpha,n)) = f((\tau,n)) \).

(i) Denote \( M = \{ n \in \omega : |f((\tau,n))| > \varepsilon \} \). If \( \alpha \in Z \) is
such that \( |M \cap \mathbb{P}_\alpha| = \omega \), then the continuity of \( f \) implies
\( |f((\alpha,\omega))| \geq \inf \{|f((\alpha,n))| : n \in \mathbb{P}_\alpha \cap M \} = \inf \{|f((\tau,n))| : n \in \mathbb{P}_\alpha \cap M \} > \varepsilon \). Therefore \( \{ \alpha \in \tau : \mathbb{P}_\alpha \cap M = \omega \} \) is \( \{ \alpha \in \tau : |f((\alpha,\omega))| > \varepsilon \} \) \( \cup S \). Since both sets on the right-hand side
are at most countable, \( M \in \mathcal{J}(\mathcal{F}) \), which was to be proved.

(ii) Denote \( N = \{ n \in \omega : |f((\tau,n))| \geq \varepsilon \} \). If \( \alpha \in Z \) is
such that \( |\mathbb{P}_\alpha \cap N| < \omega \), then \( |f((\alpha,\omega))| = \sup \{|f((\alpha,n))| : n \in \mathbb{P}_\alpha - N \} \leq \varepsilon \). Thus
\( \{ \alpha \in \tau : |f((\alpha,\omega))| > \varepsilon \} \subseteq \{ \alpha \in \tau : \mathbb{P}_\alpha \cap N = \omega \} \cup S \).
Since \( N \in \mathcal{J}(\mathcal{F}) \), the set \( \{ \alpha \in \tau : \mathbb{P}_\alpha \cap N = \omega \} \) is at most countable, hence the set \( \{ \alpha \in \tau : |f((\alpha,\omega))| > \varepsilon \} \) is at most countable, too. \( \Box \)

B. Jones machine. A well-known construction, the final
form of which is due to F.B. Jones, goes as follows [J]: Let \( T \)
be a non-normal space, denote \( A,B \subseteq T \) the closed, disjoint sets
which cannot be separated. Let \( Z = (T \times \omega) \cup \{ p \} \), where \( p \notin T \times \omega \).
The topology on \( Z \) is the usual product topology in all points
other than \( p \), the basic neighborhood of \( p \) is \( \{ p \} \cup (T \times (\omega - k)) \),
where \( k \in \omega \). Define an equivalence relation \( \sim \) on \( Z \) by
\( (x,n) \sim (y,m) \) iff either \( x \in A, y = x \) and \( n = 2k + 1, m = 2k + 2, \)
or \( x \in B, y = x \) and \( n = 2k, m = 2k + 1 \). The space \( J(T) \) is the
quotient space \( Z \) modulo \( \sim \).

The basic properties of \( J(T) \) are the following: If \( T \) is re-
gular (resp. Hausdorff, resp. \( T_1 \)), then \( J(T) \) is, but \( J(T) \) is not
completely regular, because the point \( p \) cannot be functionally

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separated from the closed set $A \times \{0\}$.

For the modified non-normal Tychonoff plank $T(\mathcal{F})$, denote $A = \{\langle \tau, n \rangle : n \in \omega \}$, $B = \{\langle \alpha, \omega \rangle : \alpha \in \tau \}$ and consider the space $J(T(\mathcal{F})) = J(\mathcal{F})$. (If necessary, we shall again denote the points of $J(\mathcal{F})$ as $p_x$ and $\langle (\alpha, n), (k_x) \rangle$. Then the following holds.

**Lemma 2.** Let $\mathcal{F} \subseteq [\omega]^\omega$ be uncountable, let $f : J(\mathcal{F}) \to \mathbb{K}$ be continuous, $f(p_x) = 0$, $\varepsilon > 0$. Then
\[
\{n \in \omega : |f(\langle (\tau, n), 0 \rangle)| > \varepsilon \} \in J(\mathcal{F}).
\]

**Proof.** There is some $k \in \omega$ such that for all $x \in \{p \cup \omega \times (\omega - k)\}/\sim$, $|f(x)| < \varepsilon /2$. Hence there is some even $j \geq k$ such that $|f(x)| < \varepsilon /2$ for all $x \in B \times \{j\}$.

Choose $\sigma' > 0$, $\sigma' < \varepsilon /2$. Since for each $x \in B \times \{j\}$, $|f(x)| < \varepsilon /2$, by Lemma 1,(i), the set $\{n \in \omega : |f(\langle (\tau, n), j \rangle)| > \varepsilon /2 \} \in J(\mathcal{F})$. Since $A \times \{j\}$ was identified with $A \times \{j - 1\}$, the set $\{n \in \omega : |f(\langle (\tau, n), j - 1 \rangle)| > \varepsilon /2 \}$ belongs to $J(\mathcal{F})$, too. Thus $\{n \in \omega : |f(\langle (\tau, n), j - 1 \rangle)| > \varepsilon /2 + \sigma' \} \in J(\mathcal{F})$, by Lemma 1, (II), the set $\{x \in \tau : |f(\langle (\alpha, \omega), j - 1 \rangle)| > \varepsilon /2 + \sigma' \}$ is at most countable. By the identification, $\{x \in \tau : |f(\langle (\alpha, \omega), j - 2 \rangle)| > \varepsilon /2 + \sigma' \}$ is countable, too, and the same holds for $\{x \in \tau : |f(\langle (\alpha, \omega), j - 2 \rangle)| > \varepsilon /2 + 2\sigma' \}$. Proceeding further, we obtain finally that $\{n \in \omega : |f(\langle (\tau, n), 0 \rangle)| > \varepsilon /2 + j\sigma' \} \in J(\mathcal{F})$, which was to be proved, as $\varepsilon /2 + j\sigma' < \varepsilon$. □

C. How to do it. The forthcoming lemma is fully proved in [S].

**Lemma 3.** There is an infinite maximal almost disjoint family $\mathcal{M} \subseteq [\omega]^{\omega}$ which admits a disjoint partition $\mathcal{M} = \mathcal{F} \cup \mathcal{G}$ such that $J(\mathcal{M}) = J(\mathcal{F}) = J(\mathcal{G})$. 

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Notice that both the collections $\mathcal{F}$, $\mathcal{G}$ must be uncountable. Suppose the contrary, let $\mathcal{F} = \{ F_n : n \in \omega \}$. Choose a countably infinite subset $\mathcal{G}' \subseteq \mathcal{G}$ and enumerate it as $\{ G_n : n \in \omega \}$. Then pick up inductively $k_n \in G_n$ such that $k_n > k_{n-1}$. Now the set $K = \{ k_n : n \in \omega \}$ belongs to $\mathcal{F}'$, for $K \cap F$ is finite for each $F \in \mathcal{F}$. On the other hand, the set $\{ M \cap K : M \in \mathcal{M} \}$ and $|M \cap K| = \omega$ is an infinite maximal almost disjoint family on $K$, hence it cannot be countable. Thus $K \in \mathcal{F}'$, $K \notin \mathcal{F}(\mathcal{M})$, which contradicts the lemma.

The spaces we promised to construct, are $X = J(\mathcal{F})$, $Y = J(\mathcal{G})$, where $\mathcal{F}$ and $\mathcal{G}$ are as in Lemma 3. Let $\tau = |\mathcal{F}|$, $\mu = |\mathcal{G}|$; using the notation as before, denote

$$\Delta = \{ (((\tau,n),0)_{\mathcal{F}} , ((\mu,n),0)_{\mathcal{G}}) : n \in \omega \}.$$

First, we shall prove that the point $(p_{\mathcal{F}}, p_{\mathcal{G}})$ is a cluster point of $\Delta$ in $CR(X) \times CR(Y)$.

Indeed, choose arbitrarily a cozero set $U$ with $p_{\mathcal{F}} \in U \subseteq J(\mathcal{F})$, and a cozero set $V$ with $p_{\mathcal{G}} \in V \subseteq J(\mathcal{G})$. By Lemma 2, $K = \{ n \in \omega : ((\tau,n)_{\mathcal{F}} \notin U \} \in J(\mathcal{F})$ and $L = \{ n \in \omega : ((\mu,n)_{\mathcal{G}} \notin V \} \in J(\mathcal{G})$. By Lemma 3, $J(\mathcal{F}) = J(\mathcal{G}) = J(\mathcal{M})$, and clearly $J(\mathcal{M})$ is a proper ideal on $\omega$, thus $\omega = K \cup L$ is infinite.

Clearly, for $n \in \omega - K \cup L$, $(((\tau,n),0)_{\mathcal{F}} , ((\mu,n),0)_{\mathcal{G}}) \in U \times V$.

Thus each neighborhood of a point $(p_{\mathcal{F}}, p_{\mathcal{G}})$ in $CR(X) \times CR(Y)$ meets $\Delta$, which was to be proved.

Second, we shall separate the point $(p_{\mathcal{F}}, p_{\mathcal{G}})$ from $\Delta$ in the space $CR(X \times Y)$.

Define a function $f:X \times Y \to \mathbb{R}$ as follows: $f((x,y)) = 1$ provided that there are $n \in \omega$, $\alpha \in \tau + 1$ and $\beta \in \mu + 1$ such that $x = ((\alpha,n),0)_{\mathcal{F}}$, $y = ((\beta,n),0)_{\mathcal{G}}$, otherwise $f((x,y)) = 0$.  

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Clearly, \( f \uparrow \Delta = 1 \), \( f((p, p')) = 0 \), thus it remains to check that \( f \) is continuous.

Pick up \((x, y)\in X \times Y\). Then there are only four non-trivial cases:

1. \( x = ((\alpha, \omega), 0)_{\gamma} \) for \( \alpha < \omega \),
   \( y = ((\beta, \omega), 0)_{\gamma} \) for \( \beta < \omega \).

\[ U = \{x\} \cup \{(\alpha, n), i\}_{\gamma} : \alpha \in \mathbb{P}_\alpha - G_{\beta}, \ i\in\{0,1\} \}, \]
\[ V = \{y\} \cup \{(\beta, n), i\}_{\gamma} : \beta \in G_{\beta} - \mathbb{P}_\alpha, \ i\in\{0,1\} \}. \]

Since \( M \) was assumed to be almost disjoint, \( (\mathbb{P}_\alpha - G_{\beta}) \cap \]
\( (G_{\beta} - \mathbb{P}_\omega) = \emptyset \), thus \( f \uparrow U \times V = 0 \).

2. \( x = ((\alpha, \omega), 0) \) for \( \alpha < \omega \),
   \( y = ((\beta, n), 0) \) for \( \beta < \omega \), \( n < \omega \).

\[ U = \{x\} \cup \{(\alpha, m), i\}_{\gamma} : m \in \mathbb{P}_\alpha - \{n\}, \ i\in\{0,1\} \}, \]
\[ V = \{y\} \cup \{(\gamma, n), 0\} : \gamma < \omega \}. \]

Then \( f \uparrow U \times V = 0 \).

3. \( x = ((\alpha, n), 0) \) for \( \alpha < \omega \), \( n < \omega \),
   \( y = ((\beta, \omega), 0) \) for \( \beta < \omega \).

This case is symmetrical to the previous one.

4. \( x = ((\alpha, n), 0) \) for \( \alpha < \omega \), \( n < \omega \),
   \( y = ((\beta, m), 0) \) for \( \beta < \omega \), \( m < \omega \).

\[ U = \{x\} \cup \{(\alpha', n), 0\} : \alpha' < \omega \}, \]
\[ V = \{y\} \cup \{(\gamma, m), 0\} : \gamma < \omega \}. \]

Then if \( f(x, y) = 0 \), which takes place if \( n = m \), we have
\( f \uparrow U \times V = 0 \), and if \( n = m \), then \( f \uparrow U \times V = 1 \).

In any case other than these just mentioned, the existence of neighborhoods \( U, V \) with \( f \uparrow U \times V = 0 \), is obvious.

Thus \( f \) is a continuous function which separates \((p, p')\) and \( \Delta \).

**Remark.** The spaces we have constructed, are regular. One
can want, moreover, that both $X$, $Y$ have a base consisting of interiors of zero sets. It suffices to start with $T(S')$ and $T(G')$ as before, but then adopt the construction described in [W] instead of Jones machine.

References


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