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A NOTE ON CONTINUITY PRINCIPLE IN POTENTIAL THEORY
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Abstract: In this note a proof is given of a continuity property of Evans-Vasilesco type for general potentials of signed measures.

Key words: potentials of signed measures, continuity principle, domination principle

Classification: 31 C 99, 31 D 05

Let $X$ be a locally compact Hausdorff topological space and let $K$ be a continuous function-kernel on $X$, i.e. an extended-real-valued positive continuous (in the wide sense) function on $X \times X$ which is finite off the diagonal $\Delta = \{[x,x]; x \in X\}$ and strictly positive on $\Delta$. Given a Radon measure $\mu \geq 0$ on $X$ we denote by

$$K\mu : x \mapsto \int_X K(x,y) \, d\mu(y)$$

its potential. Let us recall that $K$ is termed regular (cf. [4]) if it satisfies the following continuity principle:

(C) If $\mu \geq 0$ is a Radon measure with a compact support $\text{spt} \mu$ such that the restriction of $K\mu$ to $\text{spt} \mu$ is finite and continuous, then $K\mu$ is necessarily finite and continuous on the whole space $X$.

In applications one often has to consider potentials of
signed measures; given a signed Radon measure \( \nu \) with the Jordan decomposition \( \nu = \nu^+ - \nu^- \), then its potential is defined as \( K\nu = K\nu^+ - K\nu^- \) provided the difference is meaningful everywhere on \( X \). Because of possible "cancellation of discontinuities" it may happen that \( K\nu \) is finite and continuous even though \( K\nu^+ \), \( K\nu^- \) are discontinuous (cf.\([1],[10]\)). Thus the classical Evans-Vasilesco theorem does not permit the conclusion that a Newtonian potential of a signed measure \( \nu \) must be continuous everywhere provided its restriction to \( \text{spt } \nu \) is continuous. In a discussion on the occasion of the conference "5.Tagung Über Probleme und Methoden der Mathematischen Physik" (held in Karl-Marx-Stadt in May 1973) B. W. Schulze raised the question of validity of the extended Evans-Vasilesco theorem for Newtonian potentials of signed measures. Using refined tools of abstract potential theory I. Netuka was able to supply in\([10]\) a proof of the corresponding result valid for potentials on harmonic spaces satisfying the strong domination axiom (cf.\([5]\)).

It is the purpose of this note to give an elementary proof of a related continuity property of signed potentials for kernels \( K \) obeying the following domination principle:

\[ \text{(D)} \text{ If } \mu_1 \geq 0 \text{ and } \mu_2 \geq 0 \text{ are compactly supported Radon measures with finite potentials such that } K\mu_1 \leq K\mu_2 \text{ on } \text{spt } \mu_1, \text{ then } K\mu_1 \leq K\mu_2 \text{ on the whole space } X. \]

**Remark.** The classical Riesz kernel \( [x,y] \mapsto |x-y|^{-n} \) on the Euclidean space \( X = \mathbb{R}^n \) satisfies (D) provided \( 0 < 2 < n \) (cf.\([11],[7]\) and Theorem 1.29 in \([9]\)).

The reader is referred to\([6],[7],[12]\) for general investigation of potential kernels on locally compact spaces.
The following result was presented by the author in the Analysis Seminar (held in Prague in October 1975; the proof has been included in [8], p. 245).

**Theorem 1.** Let \( K \) be a strictly positive continuous function-kernel satisfying (D) and suppose that \( \nu \) is a compactly supported signed Radon measure with a finite potential \( K \nu \). If the restriction of \( K \nu \) to \( \text{spt} \nu \) is upper semicontinuous, then \( K \nu \) is upper semicontinuous on the whole space.

The proof is based on the following two known simple lemmas.

**Lemma 1.** Any continuous function-kernel \( K \) enjoying (D) is regular.

**Proof.** Cf. [7], Corollary 1.3.10 and proof of Proposition 1.3.8.

**Lemma 2.** If \( K \) is regular and \( \mu \) is a compactly supported Radon measure such that \( K \mu \) is finite on \( \text{spt} \mu \), then there exists an increasing sequence of Radon measures \( \mu_n \Rightarrow \mu \) such that the potentials \( K \mu_n \) are finite and continuous on \( X \) and converge pointwise (as \( n \uparrow \infty \)) to \( K \mu \) on \( X \).

**Proof.** Cf. Proposition 4 in Chap. II in [3] or Lemma 1.2.4 in [7].

**Proof of Theorem 1.** If \( \nu^+ \) is trivial, then \( K \nu = -K \nu^- \) is upper semicontinuous on \( X \). Assume \( \nu^+(X) > 0 \), fix \( z \in X \) and \( \varepsilon > 0 \). Lemma 2 guarantees the existence of an increasing sequence of Radon measures \( \mu_n \Rightarrow \nu^+ \) with finite continuous potentials such that

\[
0 < K \mu_n \uparrow K \nu^+ \quad \text{as} \quad n \uparrow \infty
\]

as well as the existence of a Radon measure \( \mu \) with a continuous...
potential such that

\[(2) \, \mu \leq \nu^-, \quad K(y^- \mu)(z) < \mathcal{L}K\mu_1(z) . \]

Consequently,

\[(3) \, \quad K(y^+ \mu - \mu_n) \downarrow -K(y^- \mu) \leq 0 < \mathcal{L}K\mu_1 \]

and upper semicontinuity of the restriction of \(K\nu\) to \(spt\,\nu\) implies that also the restrictions of \(K(y^+ \mu - \mu_n)\) to \(spt\,\nu\) are upper semicontinuous. In view of (3), for \(n\) large enough \(K(y^+ \mu - \mu_n) \leq \mathcal{L}K\mu_1\) on \(spt\,\nu\), which is the same,

\[(4) \, \quad K(y^+ \mu) \leq \mathcal{L}K\mu_1 + K\mu_n + Ky^- . \]

Noting that \(spt\,(y^+ \mu) \subseteq spt\,\nu\) we conclude by (D) that (4) holds everywhere on \(X\). We have by (2),(1)

\[-K\mu(z) < \mathcal{L}K\mu_1(z) - Ky^-(z) , \]

\[K\mu_n(z) \leq Ky^+(z) . \]

Hence we get for \(f = \mathcal{L}K\mu_1 - K\mu + K\mu_n\)

\[f(z) < Ky^-(z) + 2\mathcal{L}K\mu_1(z) . \]

Since \(f\) is continuous, there is a neighbourhood \(V\) of \(z\) such that

\[x \in V \Rightarrow f(x) < Ky^-(z) + 2\mathcal{L}K\mu_1(z) . \]

which together with (4) gives

\[x \in V \Rightarrow Ky^+(x) < Ky^-(z) + 2\mathcal{L}Ky^+(z) \]

and the upper semicontinuity of \(K\nu\) at \(z\) is established.

**Remark.** The above theorem may fail to hold for regular kernels not fulfilling (D) (cf. example 9 in [8], pp.246-248).

R. Wittmann (cf. [13]) has recently proposed a new approach.
to continuity properties of signed potentials which avoids kernels and works in the framework of cones of functions. His scheme may be described as follows:

Let $X$ be a locally compact Hausdorff topological space and $P$ a convex cone of non-negative continuous functions on $X$ containing a strictly positive function. Denote by $S$ the convex cone of all (finite) functions which are pointwise limits of increasing sequences in $P$. Let $Q \subset X$ be a compact set and suppose that $P_Q \subset P$ is a convex cone possessing the following property:

$$(D_Q) \quad (p \in P_Q, q \in P, p \leq q \text{ on } Q) \Rightarrow p \leq q \text{ on } X.$$ 

(Clearly, $(D_Q)$ implies the same property with any $q \in S$.) Denote by $P^*_Q$ the linear space of all functions $f$ on $X$ for which there exist sequences $\{p_n\}$, $\{q_n\}$ in $P_Q$ and an $s \in S$ such that

(i) $|p_n - q_n| \leq s \quad (n \in \mathbb{N}),$

(ii) $\lim_{n \to \infty} (p_n - q_n)(x) = f(x), \quad x \in X.$

Then the following Wittmann's theorem holds:

**Theorem 2.** Any $f \in P^*_Q$ is already continuous throughout $X$ if only its restriction to $Q$ is continuous.

This theorem can be used to get the following corollary of Theorem 1:

If $K\gamma$ is a finite non-trivial compactly supported signed potential whose restriction to $\text{spt} \gamma = Q$ is continuous, then $K\gamma$ is continuous on the whole space.

We denote by $P$ the cone of all finite continuous potentials $K\mu$ of compactly supported Radon measures $\mu \geq 0$ and by $P_Q$ the cone of all $K\mu \in P$ with $\text{spt} \mu \subset Q$. Clearly, $(D)$ implies $(D_Q)$. By Lemma 2 there are sequences $p_n \in P_Q$, $q_n \in P_Q$ with $p_n \uparrow K\gamma^+$, $q_n \uparrow K\gamma^-$, so that $|p_n - q_n| \leq K(\gamma^+ + \gamma^-) \in S$. Theorem 2 then
R. Wittmann's proof of Theorem 2 is based on an application of the Hahn-Banach theorem as employed by H. Bauer in [7]. It is perhaps of interest to note that the direct approximation technique used for the proof of Theorem 1 above may also be used to provide the following alternative of the proof of Wittmann's theorem.

**Proof.** Let \( f \) be given by (ii), where \( p_n, q_n \in P_\Sigma \) enjoy (i) for suitable \( s \in S \); we may clearly suppose that \( s \) is strictly positive on \( X \). Let us equip the space of continuous functions \( g \) on \( Q \) with the norm

\[
\| g \|_s = \inf \left\{ \lambda \geq 0 ; \ |g| \leq \lambda \text{ on } Q \right\} .
\]

The resulting normed space \( C_s(Q) \) has dual \( C_s^*(Q) \) which is represented by those signed Radon measures \( \gamma = \gamma^+ - \gamma^- \) on \( Q \), for which \( s \) is \( (\gamma^+ + \gamma^-) \)-integrable over \( Q \). The conditions (i), (ii) mean that the sequence \( \{ p_n - q_n \}_{n=1}^\infty \) converges weakly to \( f \) in \( C_s(Q) \). Consequently, there is a sequence \( \{ u_{n1}^1 \}_{n=1}^\infty \) formed by finite convex combinations of the elements \( (p_n - q_n) \) which converges to \( f \) in \( C_s(Q) \); we may thus assume that \( \| u_{n1}^1 - f \|_s < 2^{-3} \) \( (n \in \mathbb{N}) \). Applying the same reasoning to the sequence

\[
\{ p_n - q_n \}_{n=k}^\infty
\]

we get for any \( k \in \mathbb{N} \) a sequence \( \{ u_{nk}^k \}_{n=1}^\infty \) of convex combinations of elements of (5) which converges to \( f \) in \( C_s(Q) \) and satisfies

\[
\| u_{nk}^k - f \|_s < 2^{-k-2} \quad , \quad n \in \mathbb{N} .
\]

Put \( u_n = u_{nk}^k \), \( n \in \mathbb{N} \). The sequence \( \{ u_n \}_{n=1}^\infty \) converges to \( f \).
pointwise on \( X \), because \( u_k \) is a convex combination of elements of (5) and (ii) holds. It follows from (6) that

\[ |u_n - u_{n+1}|_s < 2^{-n-1} \]

whence, in view of the definition of the norm \( \| \cdot \|_s \),

\[ u_n - 2^{-n}s \uparrow f, \quad u_n + 2^{-n}s \downarrow f \quad (n \uparrow \infty) \]

on \( Q \). Since \( u_n = p_n^* - q_n^* \) for suitable \( p_n^*, q_n^* \in P_Q \), (DQ) implies that the sequence \( \{u_n - 2^{-n}s\} \) is nondecreasing on \( X \) and the sequence \( \{u_n + 2^{-n}s\} \) is nonincreasing on \( X \), so that (7) holds on \( X \). Note that, for any \( p \in P_Q \) and \( \sigma \in S \) the following implication is true:

\[ f \leq \sigma - p \quad \text{on} \quad Q \quad \Rightarrow \quad f \leq \sigma - p \quad \text{on} \quad X . \]

Indeed, the inequality \( u_n - 2^{-n}s \leq \sigma - p \) can be rewritten in the form \( p_n^* + p \leq \sigma + 2^{-n}s + q_n^* \) which, according to (DQ), holds on \( X \) whenever it holds on \( Q \). Using (7) one gets (8). Let now \( z \) be an arbitrarily fixed point of \( X \). We have by (7)

\[ u_n(z) < f(z) + 2^{-n+1}s(z) , \]

whence we conclude by continuity of \( u_n \) that for suitable neighbourhood \( V_n \) of \( z \)

\[ x \in V_n \Rightarrow u_n(x) < f(z) + 2^{-n+1}s(z) . \]

There is a sequence \( r_k \in P \) such that \( r_k \uparrow s \quad (k \uparrow \infty) \). Note that

\[ f < u_n + 2^{-n+1}s \]

on \( Q \) by (7). Since the restriction of \( f \) to \( Q \) is continuous, for sufficiently large \( k_n \)

\[ f < u_n + 2^{-n+1}r_{k_n} \]

on \( Q \), whence by (6)
\[ f \leq u_n + 2^{-n+1} r_{k_n} \] on \( X \).

We have thus by (9)

\[
x \in V_n \implies f(x) \leq f(z) + 2^{-n+1} s(z) + 2^{-n+1} r_{k_n}(x),
\]

\[
\limsup_{x \to z} f(x) \leq f(z) + 2^{-n+1} s(z) + 2^{-n+1} r_{k_n}(z) \leq f(z) + 2^{-n+2} s(z)
\]

for any \( n \in \mathbb{N} \). This proves that \( f \) is upper semicontinuous at \( z \).

Remark. Note that local compactness of \( X \) was not needed in the above proof.

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