Commentationes Mathematicae Universitatis Carolinae

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Commentationes Mathematicae Universitatis Carolinae, Vol. 25 (1984), No. 1, 171--179

Persistent URL: http://dml.cz/dmlcz/106288

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25,1 (1984)

CONVEX 4-VALENT POLYTOPES WITH PRESCRIBED TYPES OF FACES

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Abstract: Let $(p_k; k \geqq 3)$ be a sequence of nonnegative integers. A suffiencient condition for existence 4-valent convex polytope with exactly p_k k-gons for all k is given.

Key words: Convex polytope, k-gon, 4-valent vertex

Classification: 52A25

E. Jucovič [3] proved the following theorem:

If a sequence $(p_k; k \geqq 3)$ of nonnegative integers satisfies the conditions

$$\text{(i)} \qquad \sum_{k \geq 3} (4-k)p_k = 8, \text{ and}$$

$$\text{(ii)} \qquad p_4 \geqq 13 + \sum_{k \geq 5} (3k-10)p_k$$

then there exists a convex 4-valent polytope P which contains exactly p_k k-gons for all k. (Such the sequence $(p_k; k \geqq 3)$ is called 4-realizable and the polytope P is called its realization.)

This result was improved by T. C. Enns [1] who showed that (ii) can be replaced by the condition

$$p_4 \geqq 2\sum_{k \geq 5} p_k + \max\{k : p_k \neq 0\} .$$

The purpose of the present is to prove the following.

Theorem: The sequence $(p_k; k \geq 3)$ of nonnegative integers which satisfies the conditions (i) and

$$p_4 \geqq \max \{k: p_k \equiv 1 (\mathrm{mod}\ 2)\} - 2$$

is 4-realizable.

Our theorem is an improvement over previous theorems which required considerably langer values of p_4. If $\sum_{k \geq 5} p_k \leqq 1$ our boundary is the best possible.

First we prove a lemma.

Lemma 1: A planar map R with 3-edge-connected graph (without loops and multiple edges) containing exactly p_k k-gons and k-valent vertices for all k exists, iff there exists a 3-connected 4-valent planar map M(R) which contains exactly p_k k-gons for all k.

Proof: A medial map M(R) of a planar map R is formed in the following way [see 4, p.47]: To each edge of R there corresponds a vertex of M(R) and two vertices of M(R) are joined by an edge if they correspond to two edges which are incident with the same face of R and have common vertex. (In fig. 1 R is depisted by full lines and M(R) by dashed lines.) A medial map of any planar map is 4-valent.

It is easy to show that every 4-valent map is a medial map of a planar map because all faces of 4-valent planar map are regularly colourable by two colours.

Proof of our theorem: For every sequence $(p_k; k \geq 3)$

which satisfies the conditions of the theorem we shall described the construction of a planar 3-connected 4-valent map which contains exactly p_k k-gons. This reduction of the geometrical problem to a problem of existence of centain planar maps is possible by Steinitz s theorem [2, p. 33]: A graph is realizable as a convex polytope in E^3 if and only if it is planar and 3-connected.

The condition (i) is neccesary and follows from Euler's theorem.

Three cases and many subcases must be considered.

1. $p_k \equiv 0 \pmod 2$ for all $k \geqq 5$

1.1. $p_4 \equiv 0 \pmod 2$

The starting map is that of an octahedron containing configuration K formed by four triangles (fig.2). Replacing two triangles by 2(k-2) triangles, we form two of the prescribed k-gons for $k \geqq 4$. The resulting map again contains configuration K. This construction is repeated until all the required faces are formed.

1.2. $3 \leqq p_4 \equiv 1 \pmod 2$

Since the starting map (denoted by dashed lines in fig. 1) contains no configuration K, the first two k-gons are formed as indicated in fig. 3 for k=6. The resulting map contains a configuration K which is used to form (in pairs) all remaining prescribed faces.

2. $p_k \leqq 1$ for all $k \geqq 5$

2.1. $p_5 = 1$ and $p_k = 0$ for all $k \geqq 6$

The starting map (fig. 4) contains configuration K.

2.1.1. $p_4 \equiv 1 \pmod 2$

All prescribed k-gons are formed in pairs as in 1.1.

2.1.2. $p_4 \equiv 0 \pmod 2$

Before forming the required k-gons we change one quadrangle into two quadrangles as in fig. 5.

2.2. $\sum_{6 \leq k} p_k = 1$ and $p_5 = 0$

Figure 6 shows the starting map (full lines) consisting of three quadrangular regions a, b, c and six triangles. Except for 3-valent vertices A and B, all its vertices are 4-valent.

Let $p_s = 1$; the prescribed s-gon is formed from the quadrangle a by adding a path of length $2(s-4)+2$ (dashed lines) from A to B. This path is constructted in such a way that it divides b and c exclusively into quadrangles and triangles. The resulting map contains, in addition to the one s-gon, $s+4$ triangles and $s-2$ quadrangles. (Note: The map of fig. 6 is identical to the medial of the map in fig. 7.)

If $p_4 \equiv 0 \pmod s$, the remaining $p_4-(s-2)$ quadrangles are formed in pairs from configuration K_0 as in 1.

If $p_4 \equiv 1 \pmod s$, the construction step of 2.1.2. is used first.

Since in all the remaining cases the quadrangles are formed in the same way, we shall omit their creation from all following description.

2.3. $\sum_{k \geq 5} p_k = n \geq 2$

The starting map is the same as in 2.2; the construction is also similar (see fig. 8 or 9).

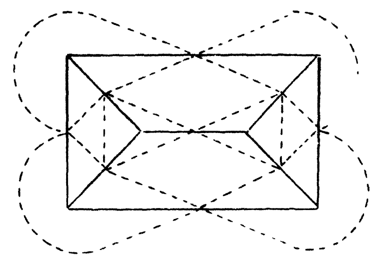

Fig 1

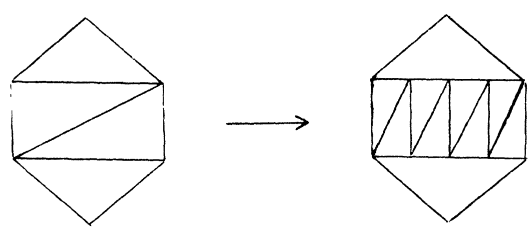

Fig 2

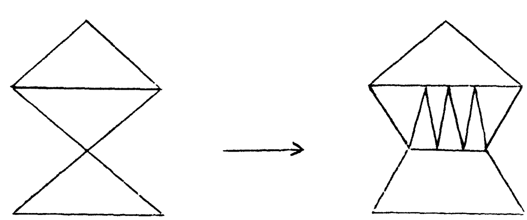

Fig 3

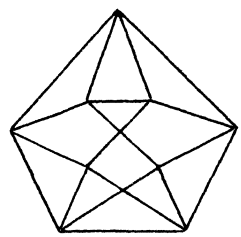

Fig 4

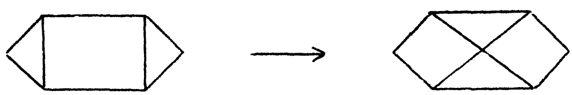

Fig 5

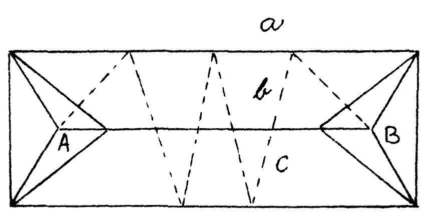

Fig 6

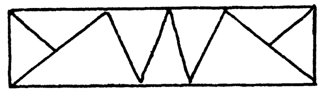

Fig. 7

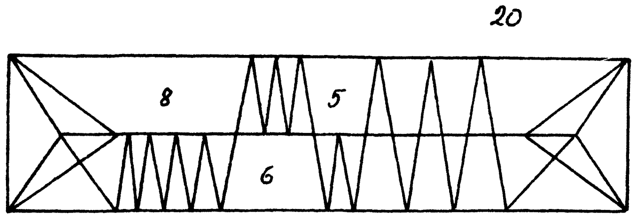

Fig. 8

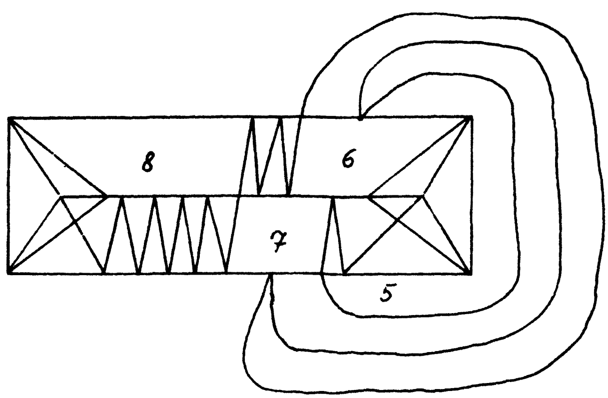

Fig 9

Let $p_{s_i} = 1$ and $p_j = 0$ for all $j \neq s_i$, $i=1,2,\ldots,n$. Moreover, let $s_i > s_k$ for $i < k$.

The construction is analogous to that of case 2.2. in that we use an additional path from vertex A to B to form the required faces. The first two faces (starting with maximum s_i) are formed concurrently in the region a and b as in fig. 8. Since when the face with fewer vertices is already finished (e.g. in b) we overlap the finishing of the remaining face (in a) with the construction of the next face (this time in c). After all the faces (except posibly one) have been formed, the construction is finished as in 2.2.

This construction which is admittedly hard to describe is illustrated in fig. 8 (for $s_1=20$, $s_2=8$, $s_3=6$, $s_4=5$) and in fig. 9 (for $s_1=11$, $s_2=8$, $s_3=7$, $s_4=6$, $s_5=5$.)

3. $2 \leqq p_k \equiv 1 \pmod 2$ for $k \geqq 5$

First we define a new sequence

$$p_k' = p_k - 2\left[\frac{p_k}{2}\right] \quad \text{for all} \quad k \geqq 5,$$

$$p_4' = \max\{k \geqq 5:\ p_k' = 1 \pmod 2\} - 2$$

$$p_3' = 8 + \sum_{k \geq 5} (k-4)p_k'.$$

This sequence satisfies the conditions of case 2. In the map containing p_k' k-gons for all k forming as it is described in case 2. we form all other prescribed k-gons, $k \geqq 5$, in pairs and $p_4 - p_4'$ quadrangles as in case 1.

This completes the proof.

<u>Lemma 2</u>: The sequence $(8, p_4, 0, 0, \ldots, 0)$ is 4-realizable if and only if p_4 is integer and $0 \leqq p_4 \neq 1$.

This is true because from the uniqueness of construction there exists no 4-valent planar 3-connected map which consists of 8 triangles and one quadrangle.

It is easy to show that the following lemma holds:

Lemma 3: Let R be a 3-edge-connected planar map having exactly one k-gon, $k \geqq 5$, and only triangles and quadranqles and every one of its vertices is of degree 3 or 4; then R contains at least two vertices which are not vertices of the k-gon.

From our theorem and lemmas 1 and 3 we have

Lemma 4: The sequence ($p_3=4+k$, p_4, $p_k=1$ for $k \geqq 5$, $p_j=0$ for all $5 \leqq j \neq k$) is 4-realizable if and only if $p_4 \geqq k-2$.

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(Oblatum 9.9. 1983)