Commentationes Mathematicae Universitatis Carolinae

Helena Zlonická

Removable singularities of solutions of the heat equation with special growth

Commentationes Mathematicae Universitatis Carolinae, Vol. 25 (1984), No. 2, 363--364

Persistent URL: http://dml.cz/dmlcz/106308

REMOVABLE SINGULARITIES OF SOLUTIONS OF THE HEAT EQUATION WITH SPECIAL GROWTH

Helena Zlonická (Charles University, Sokolovská 83, 18600 Prague, Czechoslovakia), oblatum 2.5. 1984.

Let us denote by ϱ the metric on $\mathbb{R}_{m+1}$ defined for any $x = (x_1,\dots,x_{m+1})$, $y = (y_1,\dots,y_{m+1}) \in \mathbb{R}_{m+1}$ by the formula

$$\varrho(x,y) = (|x_{m+1} - y_{m+1}| + \sum_{i=1}^{m} |x_i - y_i|^2)^{1/2}.$$

For any $q \geq 0$ we shall define set functions $\mathcal{M}^q$ and $\mathcal{H}^q$ as follows. If A is a Borel set in $\mathbb{R}_{m+1}$ then

$$\mathcal{M}^q(A) = \limsup_{\varepsilon \to 0_+} \lambda(\{x \in \mathbb{R}_{m+1} | \mathrm{dist}_\varrho(x,A) \leq \varepsilon\})/\varepsilon^{m+2-q}$$

and

$$\mathcal{H}^q(A) = \sup_{\varepsilon > 0} \inf \{\sum_{i=1}^{\infty} (\mathrm{diam}_\varrho S_i)^q | A \subset \bigcup_{i=1}^{\infty} S_i \ \&$$
$$\& \ (\forall i = 1,2,\dots : \mathrm{diam}_\varrho S_i \leq \varepsilon)\}$$

where λ denotes the Lebesgue measure in $\mathbb{R}_{m+1}$. For metric ϱ with respect to the heat equation compare [3].

Theorem 1: Let G be an open set in $\mathbb{R}_{m+1}$ and F be a relatively closed set in G. Let $0 \leq q \leq m$ and suppose f is a locally integrable function in G satisfying

$$f(x) = o(\mathrm{dist}_\varrho(x,F)^{-q}) \quad (\text{resp. } f(x) = O(\mathrm{dist}_\varrho(x,F)^{-q}))$$

as $\mathrm{dist}_\varrho(x,F) \to 0_+$ locally in G. If f satisfies (in the sense of distributions) the heat equation $(\partial/\partial x_{m+1} - \sum_{i=1}^{m} \partial^2/\partial x_i^2) f = 0$ on $G \setminus F$ and $\mathcal{M}^{m-q}(K) < +\infty$ (resp. $\mathcal{M}^{m-q}(K) = 0$) for any compact set $K \subset F$ then f satisfies the same equation on G.

Theorem 2: Let K be a compact set in $\mathbb{R}_{m+1}$ and let $0 < q \leq m$. Suppose $\mathcal{H}^{m-q}$ is not σ-finite on K (resp. $\mathcal{H}^{m-q}(K) > 0$). Then there exists a locally integrable function f on $\mathbb{R}_{m+1}$ satisfying

$$f(x) = o(\mathrm{dist}_\varrho(x,K)^{-q}) \quad (\text{resp. } f(x) = O(\mathrm{dist}_\varrho(x,K)^{-q}))$$

as $\mathrm{dist}_\varrho(x,K) \to 0_+$ such that f is a solution of the heat equation on $\mathbb{R}_{m+1} \setminus K$ but not on $\mathbb{R}_{m+1}$. Such a function f can be found as a heat potential of some non-negative Radon measure supported by K.

The proofs of both Theorem 1 and Theorem 2 are included in my thesis submitted to the Faculty of Mathematics and Physics of the Charles University in April 1984. For Theorem 1 compare the Bochner's removable singularity theorem as formulated in [2]. Note that our Theorem 1 is not implied by the Bochner's theorem. For Theorem 2 compare an analogous result of Hamann in [1] dealing with elliptic equations.

References: [1] Hamann U.: Eigenschaften von Potentialen bezüglich elliptischer Differentialoperatoren, Math. Nachr. 96(1980), 7-15.

[2] Harvey Polking: Removable singularities of solutions of linear partial differential equations, Acta Mathematica 125(1970), 39-56.

[3] Král J.: Hölder-continuous heat potentials, Accad. Naz. Lincei, Rendiconti Cl. Sc. fis., mat. Ser. VIII(1971), vol. LI, 17-19.

A CLOSED SEPARABLE SUBSPACE NOT BEING A RETRACT OF $\beta\mathbb{N}$

Petr Simon (Mathematical Institute of Charles University, Sokolovská 83, 18600 Praha, Czechoslovakia), oblatum 17.4. 1984.

D. Maharam [M] proved that the following are equivalent:

(a) For each ideal $I \subseteq \mathcal{P}(\mathbb{N})$, if there is a one-to-one homomorphism from $\mathcal{P}(\mathbb{N})/I$ to $\mathcal{P}(\mathbb{N})$, then there is a lifting from $\mathcal{P}(\mathbb{N})/I$ to $\mathcal{P}(\mathbb{N})$, too;

(b) every non-void closed separable subspace of $\beta\mathbb{N}$ is a retract of $\beta\mathbb{N}$,

and has raised the question, whether (a) or (b) is a true statement.

The answer to the Maharam's problem is in negative. We can prove the two theorems below.

Theorem 1. There exists a subspace $X \subseteq \beta\mathbb{N} - \mathbb{N}$ satisfying the following:

(1) $X = \bigcup_{n\in\omega} X_n$, where $|X_0| = 1$ and for each $n \in \omega$, the set X_n is countable discrete;

(2) for each $n < m < \omega$, $X_n \subseteq \overline{X}_m - X_m$;

(3) for each $n < \omega$ and for each $x \in X_n$, x is a $\mathfrak{c}$ - OK point in $\overline{X}_{n+1} - X_{n+1}$;

(4) suppose $\{U_k : k \in \omega\} \subseteq \mathcal{P}(\mathbb{N})$ to be a family of sets such that for some $n_0 < \omega$, $U_0^* \cap X_{n_0}$ is finite and for each $i < k < \omega$, $U_i^* \cap X_{n_0+i} \subseteq U_k^*$. Then there is a family $\{V_\alpha : \alpha \in \mathfrak{c}\} \subseteq \mathcal{P}(\mathbb{N})$ such that for each $\alpha \in \mathfrak{c}$, $V_\alpha^* \supseteq X \cap \bigcap_{k\in\omega} U_k^*$ and for each $k < \omega$ and for each finite set $\alpha_0 < \alpha_1 < \dots < \alpha_k < \mathfrak{c}$, $\bigcap_{i\le k} V_{\alpha_i}^* \subseteq \bigcap_{i\le k} U_i^*$;

(5) for each mapping $f: \mathbb{N} \to X$ there is a set $T \subseteq \mathbb{N}$ and an integer $n_1 < \omega$ such that $T^* \cap X \neq \emptyset$ and for each $n > n_1$, $X_n \cap \overline{f[T]} \cap X_{n+1} = \emptyset$.

Theorem 2. If a subspace $X \subseteq \beta\mathbb{N}$ satisfies (1) - (5) from Theorem 1, then $\overline{X}$ is not a retract of $\beta\mathbb{N}$.

It should be noted that the first example of a closed separable subspace of $\beta\mathbb{N}$ which is not a retract of $\beta\mathbb{N}$ was given by M. Talagrand under CH in [T] and the second one by A. Szymański under MA in [S].

References: [M] D. Maharam: Finitely additive measures on the integers, Sankhya, Ser. A, Vol. 38(1976), 44-59.