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A CLOSED SEPARABLE SUBSPACE NOT BEING A RETRACT OF $\beta N$

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D. Maharam [M] proved that the following are equivalent:

(a) For each ideal $I \subseteq \mathcal{P}(\mathbb{N})$, if there is a one-to-one homomorphism from $\mathcal{P}(\mathbb{N})/I$ to $\mathcal{P}(\mathbb{N})$, then there is a lifting from $\mathcal{P}(\mathbb{N})/I$ to $\mathcal{P}(\mathbb{N})$, too;
(b) every non-void closed separable subspace of $\beta \mathbb{N}$ is a retract of $\beta \mathbb{N}$, and has raised the question, whether (a) or (b) is a true statement.

The answer to the Maharam's problem is in negative. We can prove the two theorems below.

Theorem 1. There exists a subspace $X \subseteq \beta \mathbb{N} - \mathbb{N}$ satisfying the following:

1. $X = \bigcup_{n \in \omega} X_n$ where $|X_0| = 1$ and for each $n \in \omega$, the set $X_n$ is countable discrete;
2. for each $n < m < \omega$, $X_n \subseteq X_m - X_m$;
3. for each $n < \omega$ and for each $x \in X_n$, $x$ is a $\phi$ - OK point in $X_{n+1} - X_n$;
4. suppose $\{U_k : k \in \omega\} \subseteq \mathcal{P}(\mathbb{N})$ to be a family of sets such that for some $n_0 < \omega$, $U_k^* \cap X_n = \emptyset$ for each $k < \omega$, $U_k^* \cap X_n = \emptyset$. Then there is a family $\{V_{\alpha} : \alpha \in \omega^*\} \subseteq \mathcal{P}(\mathbb{N})$ such that for each $\alpha \in \omega^*$, $V_\alpha \cap X_n = U_k^*$ and for each $k < \omega$ and for each finite set $\alpha_0 < \alpha_1 < \cdots < \alpha_k < \omega$, $V_{\alpha_0} \cap \bigcap_{k < \omega} U_k^* \subseteq \bigcup_{k < \omega} U_k^*$;
5. for each mapping $f : \mathbb{N} \to X$ there is a set $T \subseteq \mathbb{N}$ and an integer $n_1 < \omega$ such that $T^* \cap X = \emptyset$ and for each $n > n_1$, $X_n \cap f[T] \cap X_{n+1} = \emptyset$.

Theorem 2. If a subspace $X \subseteq \beta \mathbb{N}$ satisfies (1) - (5) from Theorem 1, then $X$ is not a retract of $\beta \mathbb{N}$.

It should be noted that the first example of a closed separable subspace of $\beta \mathbb{N}$ which is not a retract of $\beta \mathbb{N}$ was given by M. Talagrand under CH in [T] and the second one by A. Szymanski under MA in [S].

References:

- 364 -
SHORT BRANCHES IN RUDIN-FROLÍK ORDER

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Rudin-Frolík order of types of ultrafilters in \( \beta \mathbb{N} \) has the following properties:

1. each type of ultrafilters has at most \( 2^{\mathcal{O}} \) predecessors,
2. the cardinality of each branch is at least \( 2^{\mathcal{O}} \).

Thus, in Rudin-Frolík order the cardinality of branches can be only \( 2^{\mathcal{O}} \) or \( (2^{\mathcal{O}})^+ \). It was shown in [1] that there exists a chain order - isomorphic to \( (2^{\mathcal{O}})^+ \). Hence, the existence of a branch of cardinality \( (2^{\mathcal{O}})^+ \) is proved.

The following result solves the problem of the existence of a branch having smaller cardinality.

**Theorem.** In Rudin-Frolík order there exists an unbounded chain order-isomorphic to \( \omega_1 \).

By the properties (1) and (2) the branch containing this chain has cardinality \( 2^{\mathcal{O}} \).

References:  

RESULTS ON DISJOINT COVERING SYSTEMS ON THE RING OF INT EgERS

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A system of congruence classes

\[(a_1 \pmod{n_1}, a_2 \pmod{n_2}, \ldots, a_k \pmod{n_k})\]

will be called a disjoint covering system (DCS) if for every integer \( x \) there is exactly one \( i \in \{1, 2, \ldots, k\} \) such that \( x \equiv a_i \pmod{n_i} \). The integers \( n_1, n_2, \ldots, n_k \) will be called moduli of (1) and their least common multiple will be called the common modulus of (1).

If \( k > 1 \) then no two moduli of (1) are relatively prime. This condition can be expressed in the form

\[\bigwedge_{i=1}^{k} \bigwedge_{j=1}^{k} \psi(n_i, n_j)\]

where \( \psi(x, y) \) is the formula

\[\exists z \exists u \exists v (x \neq 1 \land z \cdot u = x \land z \cdot v = y)\]

Consider more generally the formulae of the form