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Results on disjoint covering systems on the ring of integers


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SHORT BRANCHES IN RUDIN-FROLÍK ORDER

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Rudin-Frolík order of types of ultrafilters in βω has the following properties:
1. each type of ultrafilters has at most 2^ω0 predecessors,
2. the cardinality of each branch is at least 2^ω0.

Thus, in Rudin-Frolík order the cardinality of branches can be only 2^ω0 or (2^ω0)^+. It was shown in [1] that there exists a chain order - isomorphic to (2^ω0)^+. Hence, the existence of a branch of cardinality (2^ω0)^+ is proved.

The following result solves the problem of the existence of a branch having smaller cardinality.

Theorem. In Rudin-Frolík order there exists an unbounded chain order-isomorphic to ω_1.

By the properties (1) and (2) the branch containing this chain has cardinality 2^ω_0.

References:

RESULTS ON DISJOINT COVERING SYSTEMS ON THE RING OF INTEGERS

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A system of congruence classes
(1) a_1(mod n_1), a_2(mod n_2), ..., a_k(mod n_k)
will be called a disjoint covering system (DCS) if for every integer x there is exactly one i ∈ {1, 2, ..., k} such that x \equiv a_i(mod n_i). The integers n_1, n_2, ..., n_k will be called moduli of (1) and their least common multiple will be called the common modulus of (1).

If k > 1 then no two moduli of (1) are relatively prime. This condition can be expressed in the form
(2) \bigwedge_{i=1}^{k} \bigwedge_{j=1}^{k} \forall(n_1, n_j)
where \exists(x, y) is the formula
\exists z \exists u \exists v (z \neq 1 \land z \cdot u = x \land z \cdot v = y)
Consider more generally the formulae of the form
- 365 -
(3) \[ \psi(n_1, \ldots, n_r) \]

which are true for all DCS (1) with \( k > 1 \), where \( \psi(x_1, \ldots, x_r) \) is a first-order formula with the only non-logical symbol \( \psi \) for multiplying. The main result of [1] is that every such formula (3) is a consequence of (2). Hence the condition (2) is the strongest among all conditions of the form (3) which hold for all non-trivial DCS (i.e., DCS different from \( \{2\} \)). The proof uses product-invariant relations, i.e., the relations which are invariant with respect to all automorphism of the semigroup \((N, \cdot)\).

For every prime \( p \) the DCS
\[ \{0 \text{ (mod } p), 1 \text{ (mod } p), \ldots, p-1 \text{ (mod } p)\} \]
has the following property:

The union of any subset \( A \) of (4), \( 1 < \text{card}(A) < k \)

is a congruence class (by any modulus).

All DCS (except \( \{2\} \)) with this property will be called irreducible DCS. There are IDCS which are not of the form (4). For example, the congruence classes
\[ 0, 4 \text{ (mod } 6), 1, 3, 5, 9 \text{ (mod } 10), 2 \text{ (mod } 15), 7, 8, 14, 20, 26, 27 \text{ (mod } 30) \]
form an IDCS with the common modulus 30 (it is Porubsky's example of a nonnatural DCS in essential). In [2] many IDCS are constructed and it is proved that an IDCS with the common modulus \( n \) exists if and only if \( n \) is a prime (then only (4) can be obtained) or \( n \) is divisible by at least three different primes. Further, an operation of splitting is defined which allows to obtain all DCS from the degenerated DCS \( \{Z\} = \{0 \text{ (mod } 1)\} \) and the IDCS. If only IDCS of the form (4) are used then so called natural DCS are exactly obtained.

For every prime \( p \) denote \( \mathcal{F}(p) = p - 1 \), and extend the function \( \mathcal{F} \) to the set \( \mathbb{N} \) by the formula \( \mathcal{F}(x,y) = \mathcal{F}(x) + \mathcal{F}(y) \).

The Mycielski's conjecture stated \( k \geq 1 + \mathcal{F}(n_1) \) for every DCS (1) and every \( i \in \{1, 2, \ldots, k\} \). The main result of 3 is that for all DCS (which are not natural (hence e.g. for all IDCS which are not of the form (4)) it holds
\[ k \geq 6 + \mathcal{F}(n_1) \].

The proof is rather complicated but elementary. The constant 6 in (5) is the best possible. We stated the hypothesis that the modulus \( n_1 \) in (5) can be replaced by the common modulus of (1).

The IDCS with the common modul pq\( r \) (where \( p, q, r \) are distinct primes) are completely described, and the number of them is determined, in [4].

References:
The aim of this, and the subsequent note, is to announce a selection of results presented at the Colloquium on Topology held in Eger in August 1983, and at the Semester of Topology in Banach Center in April 1984. I feel that it is time to prove deeper results about Suslin sets derived from Borel sets in compact spaces.

1. By a space we mean a completely regular $T_2$ topological space. We denote by $\mathcal{G}(\mathcal{M})$ the collection of Suslin sets derived from the collection of sets $\mathcal{M}$. Recall that $\mathcal{G}(\mathcal{G}(\mathcal{M})) = \mathcal{G}(\mathcal{M})_c \supseteq \mathcal{M}_c \cup \mathcal{M}_d$. We denote by $\mathcal{G}_d(\mathcal{M})$ the sets in $\mathcal{G}(\mathcal{M})$ with disjoint Suslin representation. Denote by $\Sigma$ the space $\omega^\omega$ with product topology where $\omega$ has the discrete topology.

**Lemma 1.** Let $Y$ be a subset of a space $X$. Then
(a) $Y \in \mathcal{G}(\text{closed}(X))$ iff some closed set in $X \times \Sigma$ projects onto $Y$.
(b) $Y \in \mathcal{G}(\text{open}(X))$ iff some open set in $X \times \Sigma$ projects onto $Y$.
(c) $Y \in \mathcal{G}(\text{open}(X) \cup \text{closed}(X))$ (or $\mathcal{G}(\text{Borel}(X))$) iff the intersection of a closed set and a $G_d$ set in $X \times \Sigma$ projects onto $Y$.

Note that (a) is classical, and (c) is essentially due to Fremlin [Frel].

**Theorem 1.** The following conditions on a space $X$ are equivalent:
(a) Some Čech complete subspace of $X \times \Sigma$ projects onto $X$.
(b) If $X$ is a subspace of $Z$ then $X \in \mathcal{G}(\text{Borel}(Z))$.
(c) $X$ is obtained by Suslin operation from locally compact sets in some $Z$.
(d) There exists a complete sequence of $\sigma$-relatively open covers of $X$.

A space $X$ satisfying the equivalent conditions in Theorem 1 will be called Čech-analytic (following [Frel]). To be sure note that a cover $\mathcal{U}$ of $X$ is called $\sigma$-relatively open if $\mathcal{U} = \bigcup \{U_n | n \in \omega\}$ such that each $U_n$ is an open cover of $\mathcal{U}_n$. It was proved in [Frel] that if $X \in \mathcal{G}(\text{Borel}(X))$ for some compactification of $X$, then it holds for any compactification of $X$. Fremlin [Frel] introduced implicitly (1a) and showed the equivalence with Zolkov's definition. If the space $X$ is hereditarily Lindelöf then (1d) implies that $X$ has a complete sequence of countable covers, and hence it is $\omega$-analytic (or K-analytic in Choquet and Sneider terminology) by [EF]. The following result is a solution of a problem of Fremlin.

**Theorem 2.** A space $X$ is $\omega$-analytic iff it is Čech analytic and there exists an usco-compact correspondence from a separable metric space onto $X$.

The proof is based on the following

**Lemma 2.** Let $f$ be a perfect mapping of $X$ onto a metrizable space $Y$, and let $\{U_n\}$ be a sequence of families of open sets in $X$.

There exists a factorization $f = h \circ g$ such that $g:X \to S$, $h:S \to Y$ are perfect, $S$ is metrizable, and for each $n$
Theorem 3. The following conditions on a space $X$ are equivalent:

1. Some Čech complete subspace of $X$ injectively projects onto $X$.
2a. $X$ is a subspace of some $Z$ then $X$ is Borel ($\mathcal{S}_1$).
2b. $X$ is obtained by the disjoint Susslin operation from locally compact subsets in some $Z \supset X$.
2c. There exists a complete sequence $\{M_n | n \in \omega \}$ of covers such that each $M_n$ is an open cover of $M_n = \bigcup M_{m+1} | n \in \omega \}$ for each $n$, and if $\sigma \subseteq \Sigma$, $M_n \in \mathcal{M}_{\sigma n}$ then
   \[ \cap \{ \bigcap \{ M_n | n \in \omega \} \cap M_{\sigma n} | n \in \omega \}. \]

A space satisfying the equivalent condition in Theorem 3 will be called Čech-Luzin. Any Čech-Luzin space $X$ is absolutely bi-Suslin (Borel), and I do not know whether or not the converse holds.

The basic stability results follow easily from (1a) and the fact that any countable ($\neq 0$) power of $\mathcal{E}$ is homeomorphic to $\mathcal{E}$.


DISTINGUISHED SUBCLASSES OF ČECH-ANALYTIC SPACES

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This is a free continuation of [Prel]. Recall that if $\mathcal{F}$ is a set of families of subsets of $X$ then a family $\{X_a | a \in \Lambda \}$ in $X$ is called $\mathcal{F}$-decomposable if there exist families $\{X_a | a \in \Lambda \}$ in $\mathcal{F}$, $n \in \omega$, such that $X_a = \bigcup \{ X_{an} | n \in \omega \}$ for each $a$. So it is clear what is meant by discretely $\mathcal{F}$-decomposable. We shall call a family $\{X_a | a \in \Lambda \}$ in a topological space uniformly discrete if it is discrete in the finest uniformity inducing the topology.

A family $\{X_a | a \in \Lambda \}$ is called isolated if it is discrete in $\bigcup X_a$.

Following [Prel], if $\kappa$ is an infinite cardinal then a space $X$ is called $\kappa$-analytic (or topologically $\kappa$-analytic, abbr. $T \kappa$-analytic) if there exists an usco-compact correspondence from the metric space $\kappa^\omega$ onto $X$ such that the image of each discrete family (equivalently, discretely $\kappa$-decomposable family) is uniformly discretely (or discretely, resp.) $\sigma$-decomposable. If the values are disjoint, then the space is called $\kappa$-Luzin (or topologically $\kappa$-Luzin, resp.), and if the values are singletons or empty then we speak about point-$\kappa$-analytic etc. spaces. Analytic means $\omega$-analytic for some $\omega$, and similarly Luzin etc. The theory of analytic and Luzin spaces was developed in [Prel]. A discussion of topologically analytic spaces appeared in [J³].