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Commentationes Mathematicae Universitatis Carolinae, Vol. 25 (1984), No. 4, 635--645

Persistent URL: http://dml.cz/dmlcz/106330

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ON BOUNDED SOLUTIONS OF A LINEAR DIFFERENTIAL EQUATION WITH A NONLINEAR PERTURBATION
Bogdan RZEPECKI

Abstract: Let $E$ be a Banach space. Suppose that $f: [0, \infty) \times E \to E$ satisfies the Carathéodory conditions and some regularity condition expressed in terms of the measure of noncompactness $\kappa$. We prove the existence of bounded solutions of the differential equation $y' = A(t)y + f(t, y)$ under the assumption that the linear equation $y' = A(t)y + b(t)$ has at least one bounded solution for each $b$ belonging to a function Banach space $B_0$.

Key words: Differential equations in Banach spaces, function spaces, admissibility, measure of noncompactness.

Classification: 34G20, 34A34, 34C11.

Introduction. Throughout this paper, $J$ denotes the half-line $t \geq 0$, $E$ a Banach space with the norm $\| \cdot \|$, and $\mathcal{L}(E)$ the algebra of continuous linear operators from $E$ into itself with the induced standard norm $\| \cdot \|$.

Consider the nonlinear differential equation

\begin{equation}
(+) \quad y'(t) = A(t)y(t) + f(t, y(t)),
\end{equation}

where $t \in J$, $A(t) \in \mathcal{L}(E)$, and $f$ is an $E$-valued function defined on $J \times E$.

We are interested in the study of bounded solutions of

(+) when $f$ satisfies the Carathéodory conditions and some regularity Ambrosetti-Szufla type condition (cf. [1], [11]) expressed in terms of the measure of noncompactness $\kappa$. 

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The method used here is based on the concept of "admissibility" due to Massera and Schäffer [8]. With (+) above we shall associate the nonhomogeneous linear equation
\[(\star) \quad y'(t) = \Lambda(t)y(t) + b(t)\]
under the assumption that has at least one bounded solution for each function \(b\) belonging to a function Banach space \(B_0\).

2. Notation and preliminaries. Let \(\omega\) denote the Kuratowski's measure of noncompactness in \(E\). (The measure \(\omega(X)\) of a nonempty bounded subset \(X\) of \(E\) is defined as the infimum of all \(\varepsilon > 0\) such that there exists a finite covering of \(X\) by sets of diameter \(\leq \varepsilon\).) For properties of the Kuratowski function \(\omega\), see e.g. [3] - [6], [10].

Further, we will use the standard notations. The closure of a set \(X\), its diameter and its closed convex hull be denoted, respectively, by \(\overline{X}\), \(\text{diam} \ X\) and \(\text{conv} \ X\). If \(X\) and \(Y\) are subsets of \(E\) and \(t\), \(s\) are real numbers, then \(tx + sy\) is the set of all \(tx + sy\) such that \(x \in X\) and \(y \in Y\). For a set \(\mathcal{U}\) of mappings defined on \(X\) we write \(\mathcal{U}(t) = \{\varphi(t) : \varphi \in \mathcal{U}\}\); \(\varphi[X]\) will denote the image of \(X\) under \(\varphi\). Moreover, we use some of the notation, definitions, and results from the book of Massera-Schäffer [8] and the paper of Boudourides [2].

Let us denote:

by \(L(J,E)\) - the vector space of strongly measurable functions from \(J\) into \(E\), Bochner integrable in every finite subinterval \(I\) of \(J\), with the topology of the convergence in the mean, on every such \(I\);

by \(B(J,\mathbb{R})\) - a Banach space, provided with the norm \(\|\cdot\|_{B(\mathbb{R})}\), of real-valued measurable functions on \(J\) such that
(1) $B(J, \mathbb{R})$ is stronger than $L(J, \mathbb{R})$ (see [8], p. 35), (2) $B(J, \mathbb{R})$ contains all essentially bounded functions with compact support, and (3) if $u \in B(J, \mathbb{R})$ and $v$ is a real-valued measurable function on $J$ with $|v| \leq |u|$, then $v \in B(J, \mathbb{R})$ and $\|v\|_{B(\mathbb{R})} \leq \|u\|_{B(\mathbb{R})}$.

Let $B_0$ - the Banach space of all strongly measurable functions $u: J \to E$ such that $\|u\| \in B(J, \mathbb{R})$ provided with the norm $\|u\|_{B(E)} = \|\|u\|_{B(\mathbb{R})}\|_{\mathcal{L}(E)}$.

by $C_0$ - the Banach space of bounded continuous functions from $J$ to $E$, with the usual supremum norm.

Let $B^*(J, \mathbb{R})$ be the associate space to $B(J, \mathbb{R})$ i.e., $B^*(J, \mathbb{R})$ is the Banach space of all real-valued measurable functions $u$ on $J$ such that

$$\|u\|_{B^*(\mathbb{R})} = \sup \left\{ \int_J |v(s)u(s)|ds : v \in B(J, \mathbb{R}), \|v\|_{B(\mathbb{R})} \leq 1 \frac{3}{2} < \infty \right\}$$

We denote by $B^*(J, E)$ the Banach space of all strongly measurable functions $u: J \to E$ such that $\|u\| \in B^*(J, \mathbb{R})$ provided with the norm $\|u\|_{B^*(E)} = \|\|u\|_{B^*(\mathbb{R})}\|_{\mathcal{L}(E)}$.

We introduce the following definitions:

**Definition 1.** The pair $(B_0, C_0)$ is called admissible (cf. [8], p. 127), if for every $b \in B_0$ there exists at least one bounded solution of $(\star)$ on $J$.

**Definition 2.** Given any subinterval $I$ of $J$, we denote by $\chi_I$ the characteristic function of $I$. The space $B(J, \mathbb{R})$ is called lean (cf. [8], p. 48; [12], p. 386), if for any nonnegative function $b \in B(J, \mathbb{R})$

$$\lim_{t \to \infty} \|\chi_I(t, \infty) b\|_{B(\mathbb{R})} = 0.$$
Our result will be proved via the fixed-point theorem given below.

Denote by $C(J,E)$ the family of all continuous functions from $J$ to $E$. The set $C(J,E)$ will be considered as a vector space endowed with the topology of uniform convergence on compact subsets of $J$.

We use the following fixed-point theorem (cf. [9], Theorem 2):

Let $\mathcal{X}$ be a nonempty closed convex subset of $C(J,E)$. Let $\Phi$ be a function which assigns to each nonempty subset $X$ of $\mathcal{X}$ a nonnegative real number $\Phi(X)$ with the following properties:

1° $\Phi(X_1) \leq \Phi(X_2)$ whenever $X_1 \subset X_2$;
2° $\Phi(X \cup \{y\}) = \Phi(X)$ for $y \in \mathcal{X}$;
3° $\Phi(\text{conv } X) = \Phi(X)$;
4° if $\Phi(X) = 0$ then $X$ is compact.

Suppose that $T$ is a continuous mapping of $\mathcal{X}$ into itself and $\Phi(TX) < \Phi(X)$ for an arbitrary nonempty set $X \in \mathcal{X}$ such that $\Phi(X) > 0$. Under these hypotheses, $T$ has a fixed point in $\mathcal{X}$.

3. Result. First of all, we assume that $A \in L(J,\mathcal{L}(E))$, the pair $(B_0,C_0)$ is admissible, and $B(J,R)$ is lean.

Let $E_0$ denote the set of all points of $E$ which are values for $t = 0$ of bounded solutions of the differential equation $y' = A(t)y$. Suppose that $E_0$ is closed and has a closed complement, i.e., there exists a closed subspace $E_1$ of $E$ such that $E$ is the direct sum of $E_0$ and $E_1$.

Let $F$ be the projection of $E$ onto $E_0$, and let $U:J \to \mathcal{L}(E)$ be the solution of the equation $U' = A(t)U$ with the initial condition $U(0) = I$ (the identity mapping). For any $t \in J$ we define
a function \( G(t, \cdot) \in L(J, \mathcal{L}(B)) \) by

\[
G(t, s) = \begin{cases} 
U(t) P^{-1}(s) & \text{for } 0 \leq s < t, \\
- U(t)(I - P)U^{-1}(s) & \text{for } s \geq t.
\end{cases}
\]

Let \( G(t, \cdot) \in B^w(J, \mathcal{L}(B)) \) and \( \| G(t, \cdot) \|_{B^w(J, B)} \leq K \) for any \( t \in J \).

Moreover, let us put: \((F u)(t) = f(t, u(t)) \) for \( u \in C(J, B) \).

**Theorem.** Suppose \( f \) is a function which satisfies the following conditions:

1. For each \( x \in B \) the mapping \( t \mapsto f(t, x) \) is measurable, and for each \( t \in J \) the mapping \( x \mapsto f(t, x) \) is continuous.

2. \( \| f(t, x) \| \leq \lambda(t) \) for \((t, x) \in J \times E\), where \( \lambda \in B(J, R) \).

3. \( f \) is continuous as a map of any bounded subset of \( C(J, B) \) into the space \( B \).

Let \( g \) and \( h \) be functions of \( J \) into itself such that \( g \in B(J, R) \) with \( \sup \{ \int_J \| G(t, s) \| \, ds : t \in J \} \leq 1 \), and \( h \) is nondecreasing with \( h(0) = 0 \) and \( h(t) < t \) for \( t > 0 \). Assume in addition that for any \( \varepsilon > 0 \), \( t > 0 \) and a bounded subset \( X \) of \( E \) there exists a closed subset \( Q \) of \([0, t]\) such that \( \text{mes} ([0, t] \setminus Q) < \varepsilon \) and

\[
\alpha(x[I \times X]) \leq \sup \{ g(s) : s \in I \} \cdot h(\alpha(X))
\]

for each closed subset \( I \) of \( Q \).

Then for \( x_0 \in B_0 \) with a sufficiently small norm there exists a bounded solution \( y \) of (**) on \( J \) such that \( Py(0) = x_0 \).

**Proof.** By Theorem 4.1 of [7], there exists \( M > 0 \) such that every bounded solution of \( y' = A(t)y \) satisfies the estimate \( \| y(t) \| \leq M \| y(0) \| \) for \( t \in J \). Now, choose a positive number \( r > \| \lambda \|_{B(R)} \) and assume that \( x_0 \in B_0 \) with \( \| x_0 \| \leq \varepsilon M^{-1}(r - \| \lambda \|_{B(R)}) \).
Denote by $\mathcal{X}$ the set of all $u \in C(J,E)$ such that $\|u(t)\| \leq r$ on $J$ and

$$\|u(t_1) - u(t_2)\| \leq r \left| \int_{t_1}^{t_2} \|A(s)\| \, ds \right| + \left| \int_{t_1}^{t_2} \lambda(s) \, ds \right|$$

for $t_1, t_2$ in $J$. Define a mapping $T$ as follows: for $u \in \mathcal{X}$,

$$(Tu)(t) = U(t)x_0 + \int_J G(t,s)(Fu)(s) \, ds.$$

Let $u \in \mathcal{X}$. For $t \in J$, by the Hölder inequality ([8], Theorem 22.2), we obtain

$$\|(Tu)(t)\| \leq \|U(t)x_0\| + \int_J \|G(t,s)\| \|Fu(s)\| \, ds \leq \|A\||x_0\| + K \|\lambda\| \|B(E)\| \leq R.$$

By Theorem 2 of [2] the function $Tu$ is a bounded solution of the differential equation $y' = A(t)y + (Fu)(t)$. Hence

$$\|(Tu)(t_1) - (Tu)(t_2)\| \leq R \left| \int_{t_1}^{t_2} \|A(s)\| \, ds \right| + \left| \int_{t_1}^{t_2} \lambda(s) \, ds \right|$$

on $J$, and therefore $Tu \in \mathcal{X}$. For $u, v \in \mathcal{X}$ and $t \in J$,

$$\|(Tu)(t) - (Tv)(t)\| \leq \int_J \|G(t,s)\| \|Fu(s) - (Fv)(s)\| \, ds \leq K \|Fu - Fv\|B(E).$$

From this we conclude that $T$ is continuous as a map of $\mathcal{X}$ into itself.

Put

$$\phi(V) = \sup \{ \infty(V(t)) : t \in J \}$$

for a nonempty subset $V$ of $\mathcal{X}$. It is not hard to see that
the function $\phi$ has the properties 1° - 4° listed in Section 2. To apply our fixed-point theorem it remains to be shown that $\phi(T(V)) < \phi(V)$ whenever $\phi(V) > 0$.

Assume $V$ is a nonempty subset of $\mathbb{R}$. Fix $t > 0$ and $\varepsilon > 0$. Since $B(J, R)$ is lean, $K \| \chi_{[a, \infty)} \| B(R) < \varepsilon$ for some $a < t$. Let $\sigma = \sigma(\varepsilon) > 0$ be a number such that

$$\int_D \| G(t,s) \| \lambda(s)ds < \varepsilon$$

for each measurable $D \subset [0,a]$ with $\text{mes}(D) < \sigma$. By the Luzin theorem there exists a closed subset $Z_1$ of $[0,a]$ with $\text{mes}([0,a] \setminus Z_1) < \sigma/2$ and the function $g$ is continuous on $Z_1$.

Let $X_0 = \bigcup \{ V(s) : 0 \leq s \leq a \}$. By our comparison condition, there exists a closed subset $Z_2$ of $[0,a]$ such that $\text{mes}([0,a] \setminus Z_2) < \sigma/2$ and

$$\alpha(f[I \times X_0]) \leq \sup \{ g(s) : s \in I \} \cdot h(\alpha(X_0))$$

for each closed subset $I$ of $Z_2$.

Define: $D = D_1 \cup D_2$, $Z = [0,a] \setminus D$, where $D_i = [0,a] \setminus Z_i$ ($i = 1,2$). We have

$$\alpha(\{ \int_D G(t,s)(\mathsf{F}u)(s)ds : u \in V \}) \leq \alpha(\{ \int_D G(t,s)(\mathsf{F}u)(s)ds : u \in V \}) \leq 2 \cdot \sup \{ \| \int_D G(t,s)(\mathsf{F}u)(s)ds \| : u \in V \} \leq 2 \cdot \int_D \| G(t,s) \| \lambda(s)ds < 2 \varepsilon$$

and

$$\alpha(\{ \int_0^\infty G(t,s)(\mathsf{F}u)(s)ds : u \in V \}) \leq 2 \cdot \int_0^\infty \| G(t,s) \| \lambda(s)ds \leq 2K \| \chi_{[a, \infty)} \| B(R) \leq 2 \varepsilon.$$

Let $c_1 = \sup \{ g(s) : s \in Z_1 \}$, $c_2 = \sup \{ \| G(t,s) \| : s \in Z \}$.

Since $Z$ is compact, for any given $\varepsilon' > 0$ there exists a $\eta > 0$
such that $|s_1' - s_1''| < \eta$ with $s_1', s_1'' \in [0, t] \cap \mathbb{N}$, $|s_2' - s_2''| < \eta$ with $s_2', s_2'' \in [t, a] \cap \mathbb{N}$ and $|s' - s''| < \eta$ with $s'$, $s'' \in \mathbb{Z}$ implies $c_1 \alpha(x_0) \| G(t, s_j') - G(t, s_j'') \| < \varepsilon'$ $(j = 1, 2)$ and $c_2 \alpha(x_0) |g(s') - g(s'')| < \varepsilon'$.

Let $I_i = [t_{i-1}, t_i] \setminus D$ $(i = 1, 2, \ldots, m)$, where

$$0 = t_0 < t_1 < \ldots < t_i = t < \ldots < t_m = a$$

with $|t_i - t_{i-1}| < \eta$. We shall prove below that

$$\alpha(\bigcup \{ G(t, s) f[I_i \times X_0] : s \in I_i \}) \leq \sup \{ \| G(t, s) \| : s \in I_i \} \cdot \alpha(f[I_i \times X_0])$$

In fact, for $\varepsilon_0 > 0$ there exist a number $\eta_0 > 0$ and sets $W_j$, $j = 1, 2, \ldots, n$, such that

$$f[I_i \times X_0] = \bigcup_{j=1}^{n} W_j, \text{ diam } W_j < \varepsilon_0 + \alpha(f[I_i \times X_0])$$

and

$$\| G(t, s') - G(t, s'') \| \cdot \sup \{ \| x \| : x \in f[I_i \times X_0] \} < \varepsilon_0$$

for $s'$, $s'' \in I_i$ with $|s' - s''| < \eta_0$. Divide the interval $I_i$ into $r$ parts $d_1 < d_2 < \ldots < d_r$ in such a way that

$$|d_{k+1} - d_k| < \eta_0 \quad (k = 1, 2, \ldots, r).$$

Furthermore, let us denote by $X_{jk}$ $(j = 1, 2, \ldots, n; \ k = 1, 2, \ldots, r)$ the set of all $x \in E$ such that there exists a point $w \in W_j$ with $\| x - G(t, d_k) w \| < \varepsilon_0$.

Let $x = G(t, s_0) z_0$, where $s_0 \in [d_q, d_{q+1})$ and $z_0 \in W_p$. Then

$$\| x - G(t, s_0) z_0 \| \leq \| G(t, s_0) - G(t, d_p) \| \| z_0 \| < \varepsilon_0$$

hence $x \in X_{pq}$. Consequently,

$$\bigcup \{ G(t, s) f[I_i \times X_0] : s \in I_i \} \subset \bigcup_{j=1}^{n} \bigcup_{k=1}^{r} X_{jk}.$$ 

If $\| x_\phi - G(t, d_k) w_\phi \| < \varepsilon_0$ $(\phi = 1, 2)$ with $x_\phi \in X_{jk}$ and $w_\phi \in W_j$, then

$$\| x_1 - x_2 \| \leq \| x_1 - G(t, d_k) w_1 \| + \| G(t, d_k) w_1 - G(t, d_k) w_2 \| + - 642 -$$
Therefore,
\[
\alpha \left( \bigcup \{ G(t, s) f[I_1 \times I_0] : s \in I_1 \} \right) \leq 2 \varepsilon_0 + \varepsilon_0 + \alpha(f[I_1 \times I_0]) \cdot \sup \{ \| G(t, s) \| : s \in I_1 \},
\]
and our claim is proved.

Applying the integral mean value theorem, we get
\[
\alpha \left( \{ \int_Z G(t, s) (F_\mu)(s) \, ds : u \in V \} \right) \leq \alpha \left( \sum_{i=1}^\infty \text{mes}(I_1) \alpha \left( \bigcup \{ G(t, s)f[I_1 \times I_0] : s \in I_1 \} \right) \right) \leq \sum_{i=1}^\infty \text{mes}(I_1) \alpha \left( \bigcup \{ G(t, s)f[I_1 \times I_0] : s \in I_1 \} \right) \leq \sum_{i=1}^\infty \text{mes}(I_1) \| G(t, \sigma_1) \| g(\tau_1) h(\alpha(I_0)),
\]
where \( \sigma_1, \tau_1 \) are points in \( I_1 \) such that
\[
\| G(t, \sigma_1) \| = \sup \{ \| G(t, s) \| : s \in I_1 \} \quad \text{and} \quad g(\tau_1) = \sup \{ g(s) : s \in I_1 \}.
\]

Now, from the above, we obtain
\[
\alpha(TIV)(t) \leq \alpha \left( \{ \int_D G(t, s) (F_\mu)(s) \, ds : u \in V \} \right) + \alpha \left( \{ \int_Z G(t, s)f[I_1 \times I_0] : s \in I_1 \} \right) + \alpha \left( \{ \int^{\infty}_s G(t, s) (F_\mu)(s) \, ds : u \in V \} \right) < 4 \varepsilon + h(\alpha(I_0)) \sum_{i=1}^\infty \int_{I_1^i} \| G(t, s) \| g(s) + c_1 \| G(t, \sigma_1) - G(t, \sigma_1) \| + c_2 | g(s) - g(\tau_1) | \, ds.
\]
Suppose \( \alpha(I_0) > 0 \). From the above, it follows that
\[
\alpha(TIV)(t) < 4 \varepsilon + \sum_{i=1}^\infty \int_{I_1^i} \| G(t, s) \| g(s) + \frac{2 \varepsilon}{\alpha(I_0)} \cdot h(\alpha(I_0)) \, ds < 6\varepsilon - \delta.
\]
\[ 4\varepsilon + h(\varphi(x_0)) \int Z |G(t,s)| g(s)ds + 2\varepsilon' \cdot \text{mes } (Z). \]

Since \( V \) is almost equi-continuous and bounded, we can apply Lemma 2.2 of Ambrosetti [1] to get

\[ \varphi(x_0) = \sup \{ \varphi(V(s)) : 0 \leq s \leq a \} \leq \Phi(V). \]

Therefore

\[ \varphi(T[V](t)) < 4\varepsilon + h(\Phi(V)) \int Z |G(t,s)| g(s)ds + 2\varepsilon' \cdot \text{mes } (Z), \]

and we obtain \( \varphi(T[V](t)) \leq h(\Phi(V)) \). If \( \varphi(x_0) = 0 \), then

\[ \varphi(T[V](t)) = 0 = h(0) \leq h(\Phi(V)). \]

This proves

\[ \varphi(T[V](t)) \leq h(\Phi(V)) \text{ for each } t \in J, \]

hence \( \Phi(T[V]) \leq h(\Phi(V)) \).

The set \( \mathcal{X} \) is a closed and convex subset of \( C(J,E) \). Thus all assumptions of our fixed-point theorem are satisfied; \( T \) has a fixed point in \( \mathcal{X} \) which ends the proof.

Remark. Our result may be applied to the important case, when \( B \) is any Orlicz space \( L_\varphi \) generated by a convex \( \varphi \)-function such that \( \lim \varphi(u)/u = 0 \) and \( \lim \varphi(u)/u = \infty \).

References


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(Oblatum 21.5. 1984)