Silvia Pellegrini Manara  
On the 1-generated $s$-near-fields

*Commentationes Mathematicae Universitatis Carolinae*, Vol. 25 (1984), No. 4, 647--657


**Terms of use:**

© Charles University in Prague, Faculty of Mathematics and Physics, 1984

Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use.*

This paper has been digitized, optimized for electronic delivery and stamped with digital signature within the project DML-CZ: The Czech Digital Mathematics Library [http://project.dml.cz](http://project.dml.cz)
ON THE 1-GENERATED S-NEAR-FIELDS
S. Pellegrini MANARA

Abstract: We continue here the work [5] and study the s-near-fields $N$ that are not the sum of near-fields: they are 1-generated. Making the appropriate assumptions on the additive group we characterize the zero-symmetric case, give some examples and conclude with a characterization of the constant 1-generated s-near-fields.

Key words: Near-rings, subnear-rings, near-fields, s-near-fields, $E_n$-generated.

Classification: 12K05, 16A46

1. Introduction. In [5] we define s-near-fields, the near-rings whose proper subnear-rings are near-fields and discuss the cases in which such structures are the sum of near-fields. Here we study the s-near-fields $N$ that are not the sum of near-fields: they are 1-generated and making the appropriate assumption on the additive group, we show that, in the zero-symmetric case, $N$ is an s-near-field if and only if: $N_P$ is a near-field whose subnear-rings are near-fields, $N$ is abelian having $p^2$ exponent and each element $x \in N \setminus N_P$ generates $N$, or $N_P$ is the torsion subgroup of $N^+$ and is a near-field whose subnear-rings are near-fields and each element $x \in N \setminus N_P$ generates $N$.

We give some examples concerning the first case and in the second we prove that the additive group $N^+$ is a semi-direct sum of $N_P^+$ and a divisible group. We conclude with a characteri-
zation of the constant 1-generated s-near-fields.

2. Preliminaries. We will indicate with \( N \) a left near-ring; for the definitions and the fundamental notations we refer to [6] without an express recall.

**Definition A.** We call s-near-field a near-ring whose proper subnear-rings are near-fields.

In the following, we consider of course near-rings with proper subnear-rings.

Later on, we will say a near-ring \( N \) is \( n \)-generated if it can be generated by \( n \) elements; we will say a near-ring exactly \( n \)-generated (and we will write \( E_n \)-generated) if it has a system of \( n \) generators, but it cannot be generated by a system of \( n-1 \) elements. Moreover, for \( M \subseteq N \) we will indicate with \( \langle M \rangle \) the subnear-ring of \( N \) generated by \( M \). Remember that here the general results on the s-near-fields (see [5]) exist; however, we repeat:

**Proposition 1.** An s-near-field is at most \( E_2 \)-generated.

**Proof:** see [5] Prop. 1

**Proposition 2.** The \( N \)-subgroups and the left ideals of an s-near-field are maximal.

**Proof:** see [5] Prop. 2.

Prop. 2 allows us to extend the results of [2, 3] to our case. We start with the zero-symmetric case and examine the cases excluded in [5].

3. Zero-symmetric case

**Proposition 3.** A zero-symmetric s-near-field, if it has
nilpotent elements, is 1-generated and it is generated by each nilpotent element, which has index \( n = 2 \) and is left annihilator of \( N \).

Proof: A nilpotent element of \( N \) cannot generate a proper subnear-ring because, according to our hypotheses, it must be a near-field, thus it generates \( N \) that is in this way 1-generated (see Prop. 1 of [5]). Moreover let \( x \) be a nilpotent element of \( N \) and thus \( x^n = 0 \) for an integer \( n \). The set \( A_d(x) = \{ y \in N | xy = 0 \} \) has at least the element \( x^{n-1} \), therefore it is not null and it is a proper subnear-ring of \( N \). In our hypotheses it is a near-field, but this is excluded because it has the nilpotent element \( x^{n-1} \). Therefore \( A_d(x) = N \), \( x^2 = 0 \) and \( xN = \{ 0 \} \).

Besides:

Proposition 4. A zero-symmetric \( s \)-near-field \( N \) with nilpotent elements is:

a. without proper \( N \)-subgroups;

b. \( nN = N \iff A_d(n) = \{ y \in N | ny = 0 \} = \{ 0 \} \).

Proof a: we will show that \( \forall n \in N \) is \( nN = N \) or \( nN = 0 \) and the thesis will follow from this. Let us suppose \( \{ 0 \} \neq nN \subseteq N \), in this case \( nN \) being a proper subnear-ring is a near-field. If \( N \) has a nilpotent element \( x \), \( x \) generates \( N \) (see Prop. 3), \( x^2 = 0 \) and it is a left annihilator of \( N \), therefore \( nN \) cannot be a near-field (\( \forall n \in N \)) because the element \( nx \in nN \), if \( nx \neq 0 \), cannot belong to a near-field, but \( nx \) cannot equal 0 because this would give \( nN = \{ 0 \} \) as \( x \) generates \( N \).

Proof b: let \( n \) be an element such that \( nN = N \); if \( A_d(n) \) is proper, it is an \( N \)-subgroup of \( N \) and this is excluded from a. Viceversa, if \( A_d(n) = \{ 0 \} \), obviously \( nN = N \), thus the Proposition is proved.
Corollary 1. A zero-symmetric s-near-field, if it has nilpotent elements is:

a. N-simple, strongly monogenic, faithful and 2-primitive; moreover:

b. the semigroup \((N,\cdot)\) is the union of a right group and of \(A_s(N) = \{x \in N \mid xN = \{0\}\}\).

Proof a: it follows immediately from Prop. 4 if we recall the definitions of N-simple, strongly monogenic, faithful and 2-primitive near-ring (see [6]).

Proof b: it follows from the Theor. 4.3 of [8], if we recall the definition of a right group (see for instance [1]).

Similar results exist for the integer 1-generated s-near-fields, as we can see in Prop. 5 of [5]. In order to characterize the zero-symmetric 1-generated s-near-fields we shall first show the following:

Lemma 1. If \(N\) is a zero-symmetric strongly monogenic s-near-field, such that each of its elements has odd order, it is abelian.

Proof: if each element \(z \in N\) has odd order \(q\), each equation \(x + x = z\) has the solution \(((q+1)/2)z\). This solution is the only one: if \(x + x = y + y = z\) we have \(2x = 2y = z\). If \(r\) (odd) is the order of \(x\), \((r-1)x = (r-1)y\) (because \(r-1\) is even) and then \((r-1)x + 2x = (r-1)y + 2y\) and \(x = (r+1)y\). Therefore \((r+1)y + (r+1)y = 2y\) and thus \(2y = 0\). The order of \(y\) is also odd, thus \(ry = 0\), but \(x = (r+1)y\). It follows that \(x = y\). From the hypothesis, at least one proper subnear-ring and thus a near-field exist in \(N\), and its identity \(e\) generates a field isomorphic to \(Z_p\). Moreover from Prop. 4 it follows that such an identity \(e\) is a left identity of \(N\). In fact \(eN = N\) implies \(ex = ex\) and \(e(ex-x) = 0\),
so \( ex = x \forall x \in N \). Let \( e \) be a such left identity and \( f: N \rightarrow N \) the map so defined \( f(y) = (-e)y \forall y \in N \). This map is an automorphism of \( N^+ \) because it is obviously a homomorphism, moreover, it is a monomorphism because \( f(y) = f(y') \Rightarrow (-e)y = (-e)y' \Rightarrow (-e)y - ((-e)y) = 0 \Rightarrow (-e)y + (-e)(-y') = 0 \Rightarrow (-e)(y-y') = 0 \Rightarrow y = y' \) as \( A_d(-e) = \{0\} \) (see Prop. 4).

Lastly it is an epimorphism because \( (-e)N = N \) and therefore \( \forall z \in N \exists x \in N \mid (-e)x = z \). This automorphism is fixed point-free: if it is \( (-e)x = x \) and \( xN = N \), it is \( (-e)xy = xy \forall y \in N \), and \( -e \) is a left identity of \( N \) because the product \( xy \), while \( y \) varies in \( N \), describes \( N \) and thus \( e = -e \) and this is to be excluded because \( e \) has an odd order. If \( (-e)x = x \) and \( xN = \{0\} \), \( x \) is a generator of \( N \) and again \( (-e)z = z \forall z \in N \) and this again is absurd. In this way the hypotheses of the theorem of [4] hold and \( N \) is abelian.

We know that the identity of a proper subnear-ring of an s-near-field generates a finite field isomorphic to \( Z_p \). Therefore, from now on, we will consider as non trivial, the set of the elements of order \( p \) and more generally we will indicate \( Np^N = \{x \in N \mid p^nx = 0\} \) where \( n \) is an integer.

Lemma 2. If \( N \) is a zero-symmetric 1-generated s-near-field and if the group generated by the elements of order \( p \), \( \langle Np \rangle^+ \), is a proper subgroup of \( N^+ \), then \( \langle Np \rangle^+ \) is a left ideal of \( N \) and coincides with the set of the elements of order \( p \), \( Np \).

Proof: the group \( \langle Np \rangle^+ \) is a normal subgroup of \( N^+ \): \( \forall x \in \langle Np \rangle^+ \) it is \( x = x_1 + x_2 + \ldots + x_n \) with \( px_i = 0 \) \( \forall i \in \{1, 2, \ldots, n\} \) and then \( \forall z \in N \), \( z + x - z = z + x_1 + x_2 + \ldots + x_n - z = (z + x_1 - z) + (z + x_2 - z) + \ldots + z + x_n - z \) and
this sum belongs to \( \langle Np \rangle^+ \). Moreover, \( \langle Np \rangle^+ \) is a left ideal of \( \overline{N} \): \( \forall z \in \overline{N} \) and \( \forall x \in \langle Np \rangle^+ \), \( zx \in \langle Np \rangle^+ \). Thus \( \langle Np \rangle^+ \) is a near-field and therefore is abelian, and \( \langle Np \rangle^+ = Np \).

In the following we will indicate with \( \overline{N} \) a zero-symmetric 1-generated strongly monogenic near-ring without elements of order 2 whose additive group \( \overline{N}^+ \) is not perfect \((^0)\).

**Lemma 3.** If \( \overline{N} \) is an \( s \)-near-field in which the set \( \overline{N}p \) of the elements of order \( p \) is a proper subgroup of \( \overline{N}^+ \), then it does not have elements of order \( q \neq p \).

**Proof:** from Lemma 2 we know that if \( \langle \overline{N}p \rangle^+ \) is proper, it is a left ideal of \( \overline{N} \). Let us suppose that elements of prime order \( q \) exist in \( \overline{N} \) and let \( \langle \overline{N}q \rangle^+ \) be the subgroup generated by the elements of order \( q \). There are two cases:

1. \( \langle \overline{N}q \rangle^+ \subset \overline{N}^+ \); 2. \( \langle \overline{N}q \rangle^+ = \overline{N}^+ \).

both will be shown to be absurd.

1. **Let** \( \langle \overline{N}q \rangle^+ \subset \overline{N}^+ \). Thus \( \langle \overline{N}q \rangle^+ \) is a left ideal - see Lemma 2 - and \( \langle \overline{N}q \rangle^+ = \overline{N}q \) (set of the elements of order \( q \)). Let us consider now the near-rings generated by the identities \( e \) and \( e' \) of \( \overline{N}p \) and \( \overline{N}q \). We have \( \langle e \rangle \cong Z_p \) and \( \langle e' \rangle \cong Z_q \). If \( p < q \), let \( f \) be the following map: \( f: (\overline{N}p, \cdot) \rightarrow (\overline{N}q, \cdot) \) so defined: \( f(ne) = (ne)e' \), for \( n = 1, 2, \ldots, p-1 \). This map is a homomorphism because \( f((ne)(n'e')) = (ne)(n'e') = (ne)e'(n'e)e' = f(ne)f(n'e) \).

Therefore, in \( (\overline{N}q, \cdot) \), a subgroup \( f(Z_p, \cdot) \) isomorphic to cyclic group of order \( p-1 \), exists. Let us take now the map \( \varphi: f(Z_p, \cdot) \rightarrow (Z_q, \cdot) \) defined by \( \varphi: \langle ne \rangle e' \rightarrow ne' \) for \( n = 1, 2, \ldots, p-1 \).

\(^{(0)}\) A group is said perfect if it coincides with the derived group.
This map is a homomorphism as well, thus a subgroup of \((\mathbb{Z}_q,\cdot)\) exists but this is impossible for a prime \(p\) where \(p=2\). Elements of order 2 do not belong to \(\bar{N}\) and thus case 1 can be excluded.

2. Let \(\langle \bar{N}q \rangle^+ = \bar{N}^+\). Each element \(g \in \bar{N}\) is \(g = g_1 + g_2 + \ldots + g_n\) with \(qg_i = 0 \forall i \in \{1, 2, \ldots, n\}\). We know that \(\bar{N}^+\) is non perfect. Thus \(\bar{N}^+ = \{0\}\) or \(\bar{N}^+\) is a proper subgroup of \(\bar{N}^+\). We can easily verify that if \(\bar{N}^+\) is a proper subgroup of \(\bar{N}^+\), it is a left ideal of \(\bar{N}\) and thus \(\bar{N}^+ = \bar{N}p\) (see case 1). Therefore \(\bar{N}^+\) is abelian and \(\forall g \in \bar{N}, qg + \bar{N}^+ = q(g_1 + g_2 + \ldots + g_n) + \bar{N}^+ = (qg_1 + \bar{N}^+) + (qg_2 + \bar{N}^+) + \ldots (qg_n + \bar{N}^+) = \bar{N}^+\). Then \(\forall g \in \bar{N}, qg \in \bar{N}^+, \) and \(pqg = 0\). In this way we have shown that each element of \(\bar{N}\) has odd order and so \(\bar{N}\) must be abelian (see Lemma 1). This is absurd, it cannot be the case that \(\bar{N}^+ = \{0\}\). Anyway \(\bar{N}\) is abelian and it is a group of exponent \(q\) which cannot have elements of order \(p\). Therefore the case 2 is excluded.

Theorem 1. A near-ring \(\bar{N}\), whose additive group is not an abelian group of exponent \(p\), is an \(s\)-near-field if and only if it satisfies one of the following conditions:

1. \(\bar{N}p\) is a near-field whose subnear-rings are near-fields, \(\bar{N}\) is abelian, having a characteristic \(p^2\) and each \(x \in \bar{N} \setminus \bar{N}p\) generates \(\bar{N}\).

2. \(\bar{N}p\) is the torsion subgroup of \(\bar{N}\), is a near-field whose subnear-rings are near-fields and each element \(x \in \bar{N} \setminus \bar{N}p\) generates \(\bar{N}\).

Proof: let \(\bar{N}p\) be the set of the elements of order \(p\). Suppose that \(\langle \bar{N}p \rangle^+ = \bar{N}^+\). From the hypotheses \(\bar{N}^+\) is not perfect: if \(\bar{N}^+\) is a proper subgroup, it is also a left ideal, whose ele-
mements have prime order $q$. Thus $\mathbb{N}^+ / \mathbb{N}'$ is abelian and

$$\forall g \in \mathbb{N} \ pg + \mathbb{N}' = p(g_1 + g_2 + \ldots + g_n) + \mathbb{N}' = pg_1 + \mathbb{N}' + \ldots + pg_n + \mathbb{N}' = \mathbb{N}'$$

and $pg = 0$. Each element of $\mathbb{N}$ has odd order and so $\mathbb{N}$ is abelian (see Lemma 1). Therefore $\mathbb{N}'$ is an abelian group of exponent $p$ and this is excluded by the hypotheses. Thus $\langle \mathbb{N}_p \rangle^+$ is a proper subgroup of $\mathbb{N}^+$, and $\langle \mathbb{N}_p \rangle^+ = \mathbb{N}_p$ (see Lemma 2) and it is a left ideal, that is, a near-field whose subnear-rings are near-fields. By Lemma 3 we know that elements of prime order different from $p$ do not exist in $\mathbb{N}$.

Suppose that $\mathbb{N}$ has elements of $p$-power order; let $\mathbb{N}^n = \{ x \in \mathbb{N} | p^nx = 0 \}$. The group $\langle \mathbb{N}^n \rangle^+$ generated by $\mathbb{N}^n$ is normal in fact $\forall z \in \mathbb{N}, \forall x \in \langle \mathbb{N}^n \rangle^+, \ z + x - z = z + x_1 + x_2 + \ldots$ 

$\ldots x_n - z$ with $p^nx_1 = 0 \forall i \in \{1, 2, \ldots, n\}$. Thus $z + x - z = z + x_1 - z + (v + x_2 - z) + \ldots + (z + x_n - z)$ and this sum belongs to $\langle \mathbb{N}^n \rangle^+$. Moreover $\langle \mathbb{N}^n \rangle^+$ is a left ideal because

$\forall z \in \mathbb{N}$ and $\forall x \in \langle \mathbb{N}^n \rangle^+, \forall y \in \langle \mathbb{N}^n \rangle^+$. Then $\langle \mathbb{N}^n \rangle = \mathbb{N}^n$, but the left ideals of $\mathbb{N}$ are maximal (see Prop. 2), therefore $\mathbb{N}^n = \mathbb{N}$. \textit{Hence} $\mathbb{N}_p$ is a near-field whose subnear-rings are near-fields, $\mathbb{N}$ is abelian (see Lemma 1) and each element $x \in \mathbb{N} \setminus \mathbb{N}_p$ generates $\mathbb{N}$. We are in the case 1. If $\mathbb{N}^n = \emptyset$ for $n \geq 2$, the elements of $\mathbb{N} \setminus \mathbb{N}_p$ are torsion free and each of them generates $\mathbb{N}$; we are in case 2. Conversely, the proof is trivial.

Examples: As an additive group we consider $\mathbb{C}_9$, cyclic group of order 9 and we define the following products:
The first is an integer s-near-field, the second has nilpotent elements; they are both examples concerning case 1.

We can characterize the structure of the s-near-fields with torsion-free elements in this way:

**Theorem 2.** The additive group $\mathbb{N}^+$, of an s-near field satisfying the conditions of the case 2 of Theor. 1, is the semi-direct sum of $\mathbb{N}^p$ and of the torsion free divisible group.

**Proof:** let $\mathbb{N}^p = \{ px \mid x \in \mathbb{N}^i \}$. We observe that $\mathbb{N}^p \cap \mathbb{N}^p = \{ 0 \}$ because if $z \in \mathbb{N}^p \cap \mathbb{N}^p$ there would exist an element $y \in \mathbb{N}$ such that $py = z$ and $p^2y = 0$. But now in $\mathbb{N}$, elements of $p^2$ order do not exist. Therefore $\mathbb{N}^p$ is a proper subset of $\mathbb{N}$, and it is contained in $\mathbb{N} \setminus \mathbb{N}^p$. If $\mathbb{N}$ was abelian, $\mathbb{N}^p$ would be a left ideal of $\mathbb{N}$: in fact, $px + py = p(x + y)$ and $z(px) = p(zx) \in \mathbb{N}^p$ and so this is absurd. Thus $\mathbb{N}$ is non-abelian. Let us consider now the subgroup generated by $k\mathbb{N}$, where $k \in \mathbb{N}$. If $\langle k\mathbb{N} \rangle^+ = \{ 0 \}$, then all the elements of $\mathbb{N}$ have torsion, moreover if $\langle k\mathbb{N} \rangle^+$ is a proper subgroup of $\mathbb{N}^+$, it is a proper subnear-ring and thus a near-field. In this case $\langle k\mathbb{N} \rangle = \mathbb{N}^p$ and so this is excluded because $\mathbb{N}$ would have torsion. Lastly if $\langle k\mathbb{N} \rangle \cap \mathbb{N}^p = \{ 0 \}$, then $\langle k\mathbb{N} \rangle \subset \mathbb{N} \setminus \mathbb{N}^p$. Therefore $\langle k\mathbb{N} \rangle = \mathbb{N} \forall k \in \mathbb{N}$ and $\mathbb{N}$ is semi-divisible.

<table>
<thead>
<tr>
<th>0 1 2 3 4 5 6 7 8</th>
<th>0 1 2 3 4 5 6 7 8</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 0 0 0 0 0 0 0 0</td>
<td>0 0 0 0 0 0 0 0 0</td>
</tr>
<tr>
<td>1 0 1 2 3 4 5 6 7</td>
<td>1 0 1 2 3 4 5 6 7</td>
</tr>
<tr>
<td>2 0 8 7 6 5 4 3 2</td>
<td>2 0 8 7 6 5 4 3 2</td>
</tr>
<tr>
<td>3 0 1 2 3 4 5 6 7</td>
<td>3 0 1 2 3 4 5 6 7</td>
</tr>
<tr>
<td>4 0 8 7 6 5 4 3 2</td>
<td>4 0 8 7 6 5 4 3 2</td>
</tr>
<tr>
<td>5 0 1 2 3 4 5 6 7</td>
<td>5 0 1 2 3 4 5 6 7</td>
</tr>
<tr>
<td>6 0 8 7 6 5 4 3 2</td>
<td>6 0 8 7 6 5 4 3 2</td>
</tr>
<tr>
<td>7 0 1 2 3 4 5 6 7</td>
<td>7 0 1 2 3 4 5 6 7</td>
</tr>
<tr>
<td>8 0 8 7 6 5 4 3 2</td>
<td>8 0 8 7 6 5 4 3 2</td>
</tr>
</tbody>
</table>
From the hypotheses $\bar{N}^+$ is non perfect, and $\bar{N}$ is non abelian, so $\bar{N}'$ is a proper subgroup of $\bar{N}^+$. In particular $\bar{N}'$ is a left ideal of $\bar{N}$ and therefore $\bar{N}' = \bar{N}p$. The factor $\bar{N}^+/\bar{N}'$ is abelian, thus it is divisible and torsion-free, therefore $\bar{N}^+$ is a semi-direct sum of $\bar{N}p$ with a torsion-free divisible group (see [7] pag. 68).

At last we give a characterization of the constant 1-generated $s$-near-fields.

4. Constant case

Theorem 3. A constant near-ring $N$ is a 1-generated $s$-near-field if and only if the additive group $N^+$ is cyclic of order 4.

Proof: from Prop. 6 of [5] we know that if $N$ is a constant $s$-near-field with two ideals, it is $E2$-generated, so $N$ has only one ideal $I$. Let $\alpha$ be the non null element of the ideal $I$ isomorphic to $M_0(Z_2)$ – we recall that $M_0(Z_2) = \{ f: Z_2 \to Z_2 / f \text{ constant} \}$ (see [6] 1.4.a) – and $\alpha$ a generator of $N$. An integer $p$ such that $px = \alpha$ exist, but $2\alpha = 0$, then $|N| = |N^+| = 2p$. Moreover, the factor near-ring $N/I$ must be simple because $I$ is maximal and thus $p$ is prime. Finally, if $p \neq 2$, the group $N^+$, cyclic of order $2p$, has two proper subgroups and $N$ has two ideals, but this is excluded. Thus $p = 2$ and $N^+$ is cyclic of order 4. Conversely, the proof is trivial.

References


Facoltà di Ingegneria, Università di Brescia, Viale Europa, 39 25060 Brescia, Italy

(Oblatum 12.9. 1983)